Section 4.4: Diagonalization

Wednesday, November 16, 2022 1:12 PM

definition: a square matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

or

$$A = PDP^{-1}$$

example: we found earlier that matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

has $\lambda_1 = 2$ with $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ with $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $D = \begin{bmatrix} \overset{\lambda_1}{\boxed{2}} & 0 \\ 0 & \underset{\lambda_2}{\boxed{3}} \end{bmatrix}$ and $P = \begin{array}{cc} \vec{x}_1 & \vec{x}_2 \\ \left[\begin{matrix} 2 \\ 1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 1 \end{matrix}\right] \end{array}$

note: could also have

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

check: $A = PDP^{-1}$ where $P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \checkmark$$

Section 4.4: cont'd 2022/11/17

theorem: an $N \times N$ matrix A is diagonalizable if A has N linearly independent eigenvectors

example - a certain 3×3 matrix has two eigenvalues (one is repeated), but the repeated value has two eigenvector

- 3 eigenvectors in total

$\rightarrow$ diagonalizable

- a different 3×3 matrix also has two eigenvalues (one is repeated), and the repeated value only has one eigenvector

- 2 eigenvalues in total

$\rightarrow$ not diagonalizable

example: diagonalize $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

answer: step ① find eigenvalues

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$2 - 3\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$$\lambda = -1, 4$$

step 2 : find eigenvectors with solve $(A-\lambda I)\vec{x}=0$

for $\lambda_1 = -1$, $\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

free variable $y = t$

$\begin{cases} x = -\frac{3}{2}t \\ y = t \end{cases}$ $\quad \vec{x}_1 = t\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

so eigenvector is $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

---

now find second eigenvector

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

for $\lambda_2 = 4$ $\quad \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right]$

$\Big\downarrow$ RREF

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

let $y = t$

$\begin{cases} x = t \\ y = t \end{cases}$ $\quad$ and $\vec{x}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

step 3 : write out D and P

so $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$

(note: $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$ also acceptable)

---

now that we've written A in terms of P, D, and P^{-1}

calculate A^8

how?

$$A = PDP^{-1}$$
$$A^2 = (PDP^{-1})(PDP^{-1})$$
$$= PD P^{-1} P DP^{-1}$$
$$= PD^2P^{-1}$$
$$\Downarrow$$
$$A^N = PD^N P^{-1}$$

but $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ so $D^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$
$$\Downarrow$$
$$D^N = \begin{bmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{bmatrix}$$

so if $A = PDP^{-1}$

$$= \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

then $A^8 = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \overbrace{(-1)^8}^{=1} & 0 \\ 0 & 4^8 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4^8 \\ 2 & 4^8 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 26215 & 39321 \\ 26214 & 39322 \end{bmatrix}$$

---

example: are the following matrices diagonalizable? show your reasoning.

a) $A = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}$

answer: find eigenvalues by $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -7 & 3 \\ -1 & -1-\lambda & 1 \\ 4 & -4 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda)(-\lambda) - 28 + 12 - 3(4)(-1-\lambda) + 7\lambda + 4(1-\lambda) = 0$$

$$(1-\lambda)(1+\lambda)\lambda - 16 + 12(1+\lambda) + 7\lambda + 4(1-\lambda) = 0$$

$$\lambda - \lambda^3 \cancel{-16} \cancel{+12} + 12\lambda + 7\lambda \cancel{+4} - 4\lambda = 0$$

$$16\lambda - \lambda^3 = 0$$

$$\lambda(4-\lambda)(4+\lambda) = 0$$

$$\lambda = 0, 4, -4$$

3 distinct eigenvalues means 3 linearly independent eigenvectors, so

$\boxed{A \text{ is diagonalizable}}$

b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

answer: find eigenvalues: from $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & 2-\lambda \end{vmatrix}$$

characteristic equation $\rightarrow$ $(1-\lambda)^2(2-\lambda) = 0$

$\lambda = 1, 2$

alg mult 2 (for $\lambda = 1$), alg mult 1 (for $\lambda = 2$)

we know that $\lambda_2 = 2$ has alg mult 1 so will have one eigenvector but

$\lambda_1 = 1$ with algebraic mult 2 could have either one or two eigenvectors

one: then A is not diag

two: then A is diag

So find eigenvector(s) for the repeated eigenvalue:

for $\lambda_1 = 1$, solve $(A - \lambda I)\vec{x} = 0$

Section 4.4: cont'd
2022/11/18

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

one free variable means one eigenvector

so $\lambda_1 = 1$ has only one eigenvector (alg mult 2)
$\lambda_2 = 2$ has only one eigenvector because alg mult = 1

only two vectors in total

$\boxed{A \text{ is not diagonalizable}}$

note: if the geometric mult is equal to alg mult for each eigenvalue, then matrix is diagonalizable

---

example: Consider the matrix A where

$$\begin{cases} A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} \\ A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix} \end{cases}$$

Find matrices P and D and write A in the form $A = PDP^{-1}$.

Then calculate A.

answer: method #1. DO NOT USE! THIS METHOD IS LONG AND HORRIBLE!

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

then $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$

so $a + 2b = 3$
$c + 2d = 6$

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$$

so $a + b = 5$
$c + d = 5$

then solve that 4x4 system to get A,
now find eigenvalues and eigenvectors
to get P and D

---

method #2: USE THIS ONE!

$$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} \quad \text{so} \quad A\begin{bmatrix}1\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}$$

so $\lambda_1 = 3$ and $\vec{x}_1 = \begin{bmatrix}1\\2\end{bmatrix}$

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$$

so $\lambda_2 = 5$ and $\vec{x}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

so $P = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}$ and $D = \begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}$

then $P^{-1} = \frac{1}{-1}\begin{bmatrix}1 & -1\\-2 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix}$

$$A = PDP^{-1}$$

finally $A = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix}$

$$= \begin{bmatrix}3 & 5\\6 & 5\end{bmatrix}\begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}7 & -2\\4 & 1\end{bmatrix}$$