Section 5.1 : Orthogonality

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definition : A set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k\}$ in $\mathbb{R}^N$ is an orthogonal set iff

$$\vec{v}_i \cdot \vec{v}_j = 0 \quad \text{for all } i \neq j$$

note: $k \leq N$

examples $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthogonal set of vectors in $\mathbb{R}^3$

$\{\hat{i}, \hat{k}\}$ is an orthogonal set in $\mathbb{R}^3$

example: Is the following set of vectors an orthogonal set?

$$\left\{ \vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

answer:

$$\vec{v}_1 \cdot \vec{v}_2 = -4 + 4 + 0 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 8 + 2 - 10 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = -2 + 2 + 0 = 0$$

$\boxed{\text{yes}}$

theorem: if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k\}$ is an orthogonal set of vectors in $\mathbb{R}^N$, then these vectors are LI, linearly independent definition: an <u>orthogonal basis</u> of a subspace is a basis that is an orthogonal set.

example: $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthogonal basis of $\mathbb{R}^3$

$\left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis of $\mathbb{R}^3$

$\left\{ \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} -8 \\ 8 \end{bmatrix} \right\}$ is an orthogonal basis of $\mathbb{R}^2$

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theorem: let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k\}$ be an orthogonal basis of a subspace W. For any vector $\vec{v}$ in W, we have

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \dots c_k\vec{v}_k$$

with $c_i = \dfrac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$

digression: (will not be tested) why?

proof:
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c\vec{v}_3 + \dots c_k\vec{v}_k$$

$$\vec{v} \cdot \vec{v}_i = c_1\vec{v}_1 \cdot v_i + c_2\vec{v}_2 \cdot \vec{v}_i + \dots\dots c_k\vec{v}_k \cdot \vec{v}_i$$

recall: $v_j \cdot v_i = 0$ for $j \neq i$
orthogonal - any two vectors in basis are orthogonal

$$\vec{v} \cdot \vec{v}_i = \underbrace{c_i \vec{v}_i \cdot \vec{v}_i}$$

the only term on the right that survives has matching subscripts

$$c_i = \frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

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example: consider $B = \left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$,

which is an orthogonal basis of $\mathbb{R}^3$.
Find the components of $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$ in basis B.

answer: $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$

$$c_1 = \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{8 + 2 - 35}{16 + 4 + 25} = \frac{-25}{45} = \frac{-5}{9}$$

$$c_2 = \frac{\vec{v} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = 0$$

$$c_3 = \frac{\vec{v} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{19}{9}$$

$$\vec{v} = \frac{-5}{9} \vec{v}_1 + \frac{19}{9} \vec{v}_3$$

or $$[\vec{v}]_B = \begin{bmatrix} -5/9 \\ 0 \\ 19/9 \end{bmatrix}$$

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definition: a set of vectors in $\mathbb{R}^N$ is orthonormal

if it is an orthogonal set of unit vectors

so if $S = \{ \vec{q}_1, \vec{q}_2, \vec{q}_3, \ldots \vec{q}_k \}$ is orthonormal so if $S = \{\vec{q}_1, \vec{q}_2, \vec{q}_3, \dots \vec{q}_k\}$ is orthonormal,

then $\vec{q}_i \cdot \vec{q}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

given an orthogonal set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k\}$,

we can get an orthonormal set by scaling the vectors:

$$\left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \frac{\vec{v}_3}{\|\vec{v}_3\|}, \dots \frac{\vec{v}_k}{\|\vec{v}_k\|} \right\}$$

example: $\left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal set

Find the corresponding orthonormal set.

answer: the orthonormal set is the original set scaled by each vector's norm (or magnitude)

$$\|\vec{v}_1\| = \sqrt{16 + 4 + 25} = \sqrt{45} = 3\sqrt{5}$$

$$\|\vec{v}_2\| = \sqrt{1 + 4} = \sqrt{5}$$

$$\|\vec{v}_3\| = \sqrt{4 + 1 + 4} = 3$$

orthonormal set is $\left\{ \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

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why are orthonormal sets useful?

transformation of coordinate systems

if $\{\hat{q}_1, \hat{q}_2, \hat{q}_3, \dots \hat{q}_k\}$ is an orthonormal basis of W, then for all $\vec{v}$ in W, we can write

$$\vec{v} = c_1 \hat{q}_1 + c_2 \hat{q}_2 + c_3 \hat{q}_3 + \dots c_k \hat{q}_k$$

where $c_i = \vec{v} \cdot \hat{q}_i$ for $i = 1, 2, 3, \dots k$

why? $\hat{q}_i \cdot \hat{q}_i = 1$, since vectors q_i are all unit vectors

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orthogonal matrices:

definition: an orthogonal matrix is one whose columns form an orthonormal set

orthogonal matrix = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

each column is orthogonal to every other column

AND every column is a unit vector

cool property: iff Q is an orthogonal matrix, then

$$Q^{-1} = Q^T$$

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digression: will not be tested

each column is a unit vector

proof: for the matrix $Q = [\hat{q}_1, \hat{q}_2, \hat{q}_3 \dots \hat{q}_k]$

$$Q^T Q = \begin{bmatrix} \hat{q}_1 \\ \hline \hat{q}_2 \\ \hline \hat{q}_3 \\ \vdots \\ \hat{q}_k \end{bmatrix} \left[\hat{q}_1 \,\middle|\, \hat{q}_2 \,\middle|\, \hat{q}_3 \,\middle|\, \cdots \,\middle|\, \hat{q}_k \right]$$

$$= \begin{bmatrix} \hat{q}_1 \cdot \hat{q}_1 & \hat{q}_1 \cdot \hat{q}_2 & \hat{q}_1 \cdot \hat{q}_3 & \cdots & \hat{q}_1 \cdot \hat{q}_k \\ \hat{q}_2 \cdot \hat{q}_1 & & & & \\ & & & & \\ \hat{q}_k \cdot \hat{q}_1 & & & & \hat{q}_k \cdot \hat{q}_k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & & \\ 0 & & 1 & & & \\ 0 & & & 1 & & \\ 0 & & & & \ddots & 1 \end{bmatrix}$$

$$Q^T Q = I$$

$\uparrow$

so must also be Q^{-1}

Section 5.1: cont'd

example: is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ orthogonal?

answer: if A is orthogonal, then $A^T = A^{-1}$, and $A^T A = I$

$$A^T A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{yes}}$

example: is $R(\theta)$ in $\mathbb{R}^2$ orthogonal?

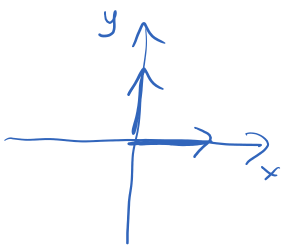
answer:

$$R^T(\theta)\, R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

example: consider the transformation in $\mathbb{R}^2$ which reflects across the y-axis. Is the associated matrix A orthogonal?



$$A\hat{\imath} = -\hat{\imath} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad A\hat{\jmath} = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

method #1: $A^TA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

method #2: $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_1 = 1$ ← unit vector

$\vec{v}_1 \cdot \vec{v}_2 = 0 \quad \leftarrow \quad \vec{v}_1 \perp \vec{v}_2$

$\vec{v}_2 \cdot \vec{v}_2 = 1$ ✓

method #3: $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

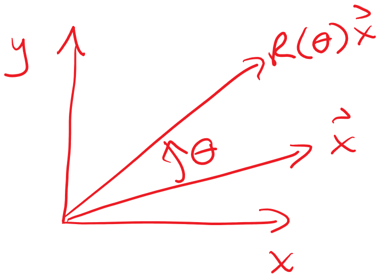
$= \frac{1}{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = A^T$ ✓

$\boxed{\text{yes}}$

orthogonal matrices are "length-preserving" transformations

- they even preserve dot products

example: $R(\theta)$ we found to be an orthogonal matrix:



the rotated vector is the same length as the original

property: the eigenvalues of an orthogonal matrix Q satisfy

$$|\lambda| = 1$$

digression: proof:

$$Q\vec{x} = \lambda \vec{x}$$

$$\|Q\vec{x}\| = |\lambda| \, \|\vec{x}\|$$

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$$\| Q\vec{x} \| = |\lambda| \, \|\vec{x}\|$$

Q may rotate/reflect vector, but does not change length

$$\| \vec{x} \| = |\lambda| \, \|\vec{x}\|$$

$$|\lambda| = 1$$

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example: show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ satisfies $|\lambda| = 1$

answer: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

recall: the absolute value of a complex number is the distance from the origin