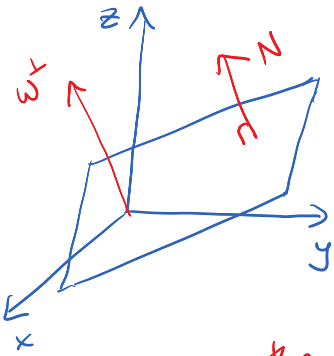
# Section 5.2: Orthogonal Complements and Projections

Tuesday, November 22, 2022 2:02 PM

Let us consider a plane $W$ which is a subspace of $\mathbb{R}^3$ and vector $\vec{N}$ which is orthogonal to plane $W$




← plane $W$ which goes through the origin

the line through the origin parallel to $\vec{N}$ is also a subspace and we write it as

$$W^{\perp}$$

definition: Let $W$ be a subspace. A vector $\vec{v}$ is orthogonal to $W$ if $\vec{v}$ is orthogonal to every vector in $W$.

The set of <u>all</u> vectors orthogonal to $W$ is called the <u>orthogonal complement</u> to $W$ and is denoted by

$W^{\perp}$ ("W perp")

if you like, $W^{\perp} = \{ \vec{v} \mid \vec{v} \cdot \vec{w} = 0$ for all $\vec{w}$ in $W \}$

example: $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; 3x + 5y + 2z = 0 \right\}$

the subspace $W$

the set of all 3D

such that

coordinates satisfy this equation

vecters

Find $W^{\perp}$.

answer: $W$ is a plane through the origin in $\mathbb{R}^3$

now the normal $N = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ is a vecter perpendicular to $W$

so $W^{\perp} = \text{span}\left(\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}\right)$ - $W^{\perp}$ is a line through the origin

note: $(W^{\perp})^{\perp} = W$

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recall: consider matrix $A$

we previously learned about the row space $\text{Row}(A)$ and the column space $\text{Col}(A)$

but what is the space of vecters orthogonal to $\text{Row}(A)$? to $\text{Col}(A)$?

for all vecters in $[\text{Row}(A)]^{\perp}$, that's equivalent to saying all of these vecters are $\perp$ to all rows of $A$.

example: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

if $\vec{x}$ is $\perp$ to all rows of $A$, then

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which means that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

what is this set of vectors, then, that you get by solving $A\vec{x} = 0$?

null space! $\mathrm{Null}(A)$

Conclusion: $[\mathrm{Row}(A)]^{\perp} = \mathrm{Null}(A)$

but what about the column space, $\mathrm{Col}(A)$?

replace $A$ by $A^T$

then $\mathrm{Row}(A^T) = \mathrm{Col}(A)$

and $[\mathrm{Col}(A)]^{\perp} = \mathrm{Null}(A^T)$

summary:

$$\left.\begin{array}{l} \mathrm{Row}(A) \perp \mathrm{Null}(A) \\ \mathrm{Col}(A) \perp \mathrm{Null}(A^T) \end{array}\right\} \text{4 fundamental spaces of an } m \times n \text{ matrix}$$

example: let $W = \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4)$

where $\vec{w}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, $\vec{w}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{w}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Find $W^\perp$.

answer: If the columns of matrix $A$ are made up of the vectors $\vec{w}_1$ to $\vec{w}_4$, then we want

$$[\text{Col}(A)]^\perp$$

and looking at the four fundamental spaces, we see that

$$[\text{Col}(A)]^\perp = \text{Null}(A^T)$$

$$A^T = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

how do we find $\text{Null}(A^T)$?

solve $A^T\vec{x} = 0$

RREF

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

let $x_2 = s$ let $x_5 = t$

$x_1 + x_2 + x_5 = 0$

$x_3 + x_5 = 0$

$x_4 = 0$

$$\begin{cases} x_1 = -s - t \\ x_2 = s \\ x_3 = -t \end{cases}$$

or $\vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

$$\begin{cases} x_2 = s \\ x_3 = -t \\ x_4 = 0 \\ x_5 = t \end{cases} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

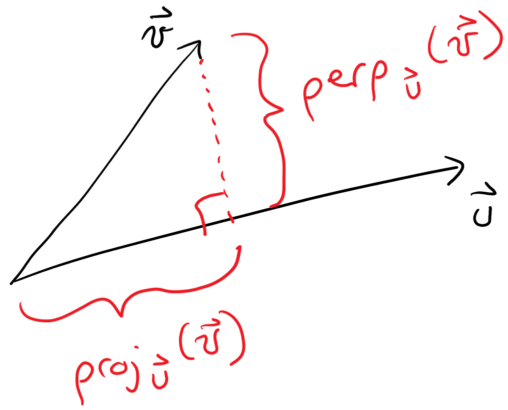
so $W^{\perp} = \text{Null}(A^T)$

$$= \text{span}\left( \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

note: because there are only 3 leading variables, $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ spans $W$ but does not form a basis

orthogonal projections:

recall: projecting $\vec{v}$ onto $\vec{u}$




$$\vec{v} = \text{proj}_{\vec{u}}(\vec{v}) + \text{perp}_{\vec{u}}(\vec{v})$$

$\text{proj}_{\vec{u}}(\vec{v})$: component parallel to $\vec{u}$

$\text{perp}_{\vec{u}}(\vec{v})$: component perpendicular to $\vec{u}$

now let's project onto a subspace:

let's say that $W$ is a plane:

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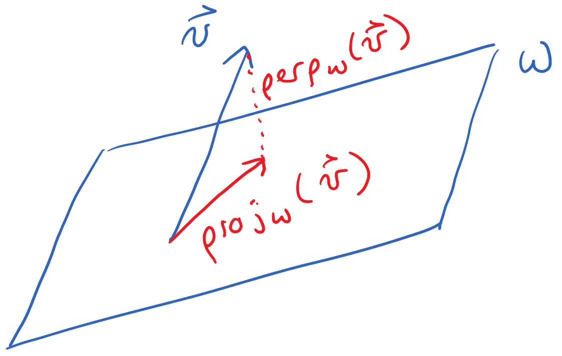



note: this is drawn in $\mathbb{R}^3$, but applies in $\mathbb{R}^N$

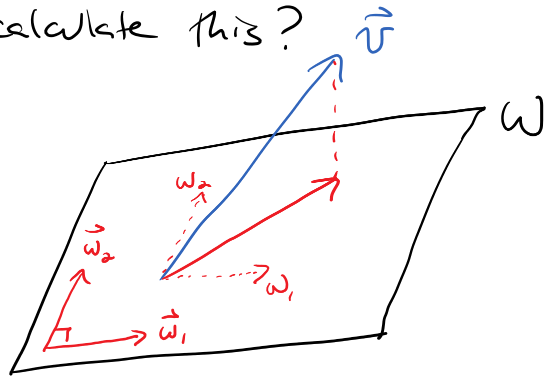
then

$$\vec{v} = proj_W(\vec{v}) + perp_W(\vec{v})$$

any vector $\vec{v}$ in $\mathbb{R}^N$ can be expressed uniquely in this way

so, how do we calculate this?

in $\mathbb{R}^3$:




$\vec{w}_1$ and $\vec{w}_2$ form an orthogonal basis for $W$

then $proj_W(\vec{v}) = proj_{\vec{w}_1}(\vec{v}) + proj_{\vec{w}_2}(\vec{v})$

subspace — first vector — second vector

but only if $\vec{w}_1 \perp \vec{w}_2$

now how do you get $perp_W(\vec{v})$?

$$\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v})$$

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so the previous example was in $\mathbb{R}^3$

in general for $\mathbb{R}^N$

$$\begin{cases} \text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v}) + \dots \text{proj}_{w_k}(\vec{v}) \\ \text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}) \end{cases}$$

if $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_k\}$ is an orthogonal basis

example: let $W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$

Find $\text{proj}_W(\vec{v})$ for $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$.

answer: let $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

observe that $\vec{w}_1 \cdot \vec{w}_2 = 0$ so $\vec{w}_1 \perp \vec{w}_2$

then $\text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v})$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{v}}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2$$

$$\boxed{\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}}$$

$$= \frac{2+1-5}{1+1+1}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{0+1+5}{0+1+1}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{-2}{\ } \begin{bmatrix} 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \end{bmatrix}$$

$$= \frac{-2}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2/3\\7/3\\11/3\end{bmatrix}$$

this is the component of $\vec{v}$ in $W$

DO NOT SCALE!

Find $perp_W(\vec{v})$.

$$perp_W(\vec{v}) = \vec{v} - proj_W(\vec{v})$$

$$= \begin{bmatrix}2\\1\\5\end{bmatrix} - \begin{bmatrix}-2/3\\7/3\\11/3\end{bmatrix}$$

$$= \begin{bmatrix}8/3\\-4/3\\4/3\end{bmatrix}$$

this is the component of $\vec{v}$ not in $W$

DO NOT SCALE!

optional quick check:

$$perp_W(\vec{v}) \cdot \vec{w}_1 = \begin{bmatrix}8/3\\-4/3\\4/3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\end{bmatrix} = 0 \quad \checkmark$$

$$perp_W(\vec{v}) \cdot \vec{w}_2 = \begin{bmatrix}8/3\\-4/3\\4/3\end{bmatrix} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix} = 0 \quad \checkmark$$