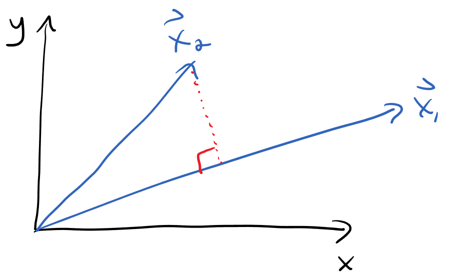
Section 5.3: Gram-Schmidt Process

Thursday, November 24, 2022 9:57 AM

this process is used to find an orthogonal basis of subspace W

begin with a basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3 \dots \vec{x}_k\}$ of W and "orthogonalize" it one vector at a time using projections

here's the idea in $\mathbb{R}^2$:



$\{\vec{x}_1, \vec{x}_2\}$ is a basis of $\mathbb{R}^2$

we call our orthogonal $\{\vec{v}_1, \vec{v}_2\}$

let $\vec{v}_1 = \vec{x}_1$

$$\vec{v}_2 = \vec{x}_2 - \underbrace{\text{proj}_{\vec{v}_1}(\vec{x}_2)}_{\text{the part of } \vec{x}_2 \text{ that is parallel to } \vec{v}_1}$$

note: if you want an orthonormal basis, then divide each vector by its norm

Gram-Schmidt process:

start from a basis $\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_k\}$ of W

then $\vec{v}_1 = \vec{x}_1$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$$

$$\vdots$$

$$\vec{v}_k = \vec{x}_k - \text{proj}_{\vec{v}_1}(\vec{x}_k) - \text{proj}_{\vec{v}_2}(\vec{x}_k) \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{v}_k)$$

finally, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is an orthogonal basis of W

(if needed, normalize vectors – make them unit vectors – to get an orthonormal basis)

Section 5.3: cont'd

example: Apply the Gram-Schmidt process to transform the following vectors into an orthonormal basis.

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

answer: let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

then
$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$
$$= \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1-2}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3/2 \\ 2 \\ 3/2 \end{bmatrix} \xrightarrow[\text{scale}]{} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$$
$$= \vec{x}_3 - \frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1+1}{1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3+4-3}{9+16+9} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{17} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6/17 \\ 9/17 \\ -6/17 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\}$ are an orthogonal basis

and $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{34}} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{17}} \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\}$ is the associated orthonormal basis

Section 5.3: cont'd

example: Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

answer: let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

then choose <u>any</u> other two vectors in $\mathbb{R}^3$ that are LI with $\vec{v}_1$

because we are lazy, we will go with

$\vec{x}_2 = \hat{i} = [\ 1$ ... $]$ and $\vec{x}_3 = \hat{j} = [\ 0$ ... $]$

$$\vec{x}_2 = \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{x}_3 = \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

note: even with laziness, can still choose $\hat{j}$ and $\hat{k}$ or $\hat{i}$ and $\hat{k}$

$\Rightarrow$ answer will not be unique

Gram - Schmidt:

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$

$$= \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{1+4+9} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{14} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 13/14 \\ -1/7 \\ 3/14 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 13 \\ -2 \\ 3 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$$

$$= \vec{x}_3 - \frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{14} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \frac{-2}{169+4+9} \begin{bmatrix} 13 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \frac{1}{91} \begin{bmatrix} 13 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \vdots \\ -3 \end{bmatrix} \qquad \begin{bmatrix} \vdots \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 9/13 \\ 6/13 \end{bmatrix} \xrightarrow{\text{Scale}} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

So orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 13 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}$

QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, it can be expressed as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix

To find the QR factorization, apply the Gram - Schmidt process to the columns of A to get the orthogonal vectors $\hat{q}_1, \hat{q}_2, \dots \hat{q}_n$

then $Q = \begin{bmatrix} \hat{q}_1 & | & \hat{q}_2 & | & \cdots & \hat{q}_n \end{bmatrix}$

and find R by $Q^T A$.

example: Find a QR decomposition of

$$A = \begin{array}{c} \begin{array}{ccc} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \end{array}$$

answer: let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$

$$= \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-2}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$$

$$= \vec{x}_3 - \frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1+1-1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{1+1+1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1/2 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

then need to make them into unit vectors to get an orthonormal set:

$$\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and $Q = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$

then finally

$$R = Q^T A = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

if you want to do an optional check, ask:

does QR in fact equal A?

optional fact:

to test if a set of vectors is orthogonal:

$$\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k \}$$

method #1: $\vec{v}_1 \cdot \vec{v}_2 = 0$

$v_1 \cdot v_3 = 0$

$\vdots$

$v_i \cdot v_j$ for all $i \neq j$

test all the pairs

method #2:

write your vectors as row vectors in matrix $\rightarrow$

$$\begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vdots \\ \vec{v}_k^{\,T} \end{bmatrix} \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \vec{v}_3 \,\middle|\, \cdots \; \vec{v}_k \right] = \begin{bmatrix} * & & & & & 0 \\ & * & & & & \\ & & * & & & \\ & & & * & & \\ & & & & * & \\ 0 & & & & & * \end{bmatrix}$$

non zero on diagonal

note: the larger the set, the better this looks