Section 5.4: Orthogonal diagonalization of symmetric matrices

Thursday, December 01, 2022 9:48 AM

If we wish to diagonalize

$$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

we can first observe that $\underbrace{A^T = A}$.

matrices with this property are called symmetric

symmetric matrices have some nice properties
- one of them is that symmetric matrices always have real eigenvalues

now, the eigenvectors corresponding to distinct eigenvalues for any matrix are LI

example: $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\qquad \lambda_1 = 5$ with $\vec{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

not symmetric $\qquad \lambda_2 = -1$ with $\vec{x_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

but the eigenvectors corresponding to distinct eigenvalues for symmetric matrices are also orthogonal

example: $C = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ $\qquad \lambda_1 = -3$ with $\vec{x_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

symmetric $\qquad \lambda_2 = 2 \qquad \vec{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x_1} \perp \vec{x_2}$

definition : a square matrix is orthogonally diagonalizable if there exists an orthogonal matrix Q and diagonal matrix D such that

$$Q^T A Q = D$$

or. $$A = QDQ^T$$

recall that $Q^T = Q^{-1}$ for orthogonal matrices

matrix with orthonormal columns

why is this cool? previously, we had $A = PDP^{-1}$

for 3×3 and above, finding P^{-1} is annoying, but taking the transpose is straightforward

spectral theorem : if A is orthogonally diagonalizable, then A is symmetric

why spectral? bright lines in the spectrum of light from elements correspond to the eigenvalues of a certain operator

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back to eigenvectors and eigenvalues for symmetric matrices

- if eigenvalues are distinct, then eigenvectors are orthogonal

- if eigenvalues are repeated, can only diagonalize if algebraic mult = geometric mult

okay, what if that's true? and the eigenvectors from repeated eigenvalues aren't orthogonal?

$\Rightarrow$ you <u>make</u> them orthogonal

by Gram-Schmidt

example: find orthogonal matrix Q and diagonal matrix D such that $Q^TAQ = D$ for

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

which has eigenvalues 2, 2, 8.

answer: find eigenvectors

$\lambda_1 = 2$ solve $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $y = s$ let $z = t$

$$\begin{cases} x = -s - t \\ y = s \\ z = t \end{cases} \qquad \vec{x}_1 = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are these orthogonal? NO! we'll have to Gram-Schmidt

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now do $\lambda_2 = 8$: solve $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \overset{RREF}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $z = t$

$$\begin{cases} x = t \\ y = t \\ z = t \end{cases} \qquad \vec{x}_2 = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

summary: for $\lambda_1 = 2$

eigenvectors $\underset{\vec{x}_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$

not orthogonal

$\lambda_2 = 8$

eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

then do Gram-Schmidt:

let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$

$$= \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

so $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\|\vec{v}_1\| = \sqrt{2}$, $\|\vec{v}_2\| = \sqrt{6}$, $\|\vec{v}_3\| = \sqrt{3}$

so $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

with $$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$