Complex Numbers

Tuesday, November 01, 2022 1:58 PM

definition: $i = \sqrt{-1}$

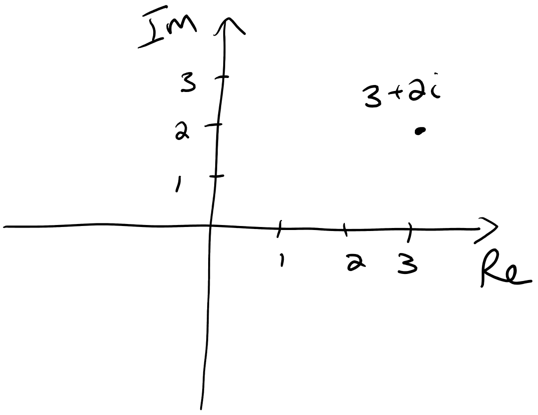
and $i^2 = -1$

note: often written as j in electronics

then complex numbers can be written in the form

$$z = \underbrace{a}_{\text{real part}} + \underbrace{bi}_{\text{imaginary part}}$$

where a and b are real



complex plane

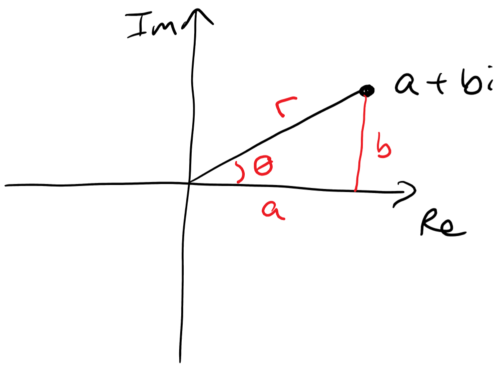
note: addition: $(3+5i) + (2-4i) = 5 + i$

add real parts, add imag. parts

multiplication:

$$(3+5i)(2+4i)$$
$$= 6 + 12i + 10i + 20i^2$$
$$= 6 + 22i + 20(-1)$$
$$= -14 + 22i$$

we can plot complex numbers in $\mathbb{R}^2$ by



we can specify this complex number in

rectangular form $a+bi$

but we can also use polar form and describe the complex number in terms of r and θ

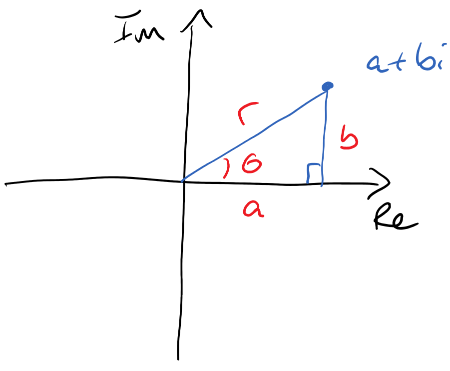
where $r^2 = a^2 + b^2$

$r = \sqrt{a^2+b^2} = |z|$

and $\tan\theta = \dfrac{b}{a}$

note: $\arctan\left(\dfrac{b}{a}\right)$ only gives values in Quadrants I and IV, so to get values in QII and III, add 180°

Complex Numbers cont'd



$b = r\sin\theta$

$a = r\cos\theta$

then $z = \underbrace{a + bi}$ can be expressed also rectangular form

in polar form (in terms of r and θ):

$$z = r\cos\theta + i r\sin\theta$$

$$= r(\cos\theta + i\sin\theta) \quad \leftarrow \text{trig form}$$

or $z = re^{i\theta} \quad \leftarrow$ exponential form

must be in radians

or $z = r\angle\theta \quad \leftarrow$ angle form

in degrees

where, generally, $\theta \in (-\pi, \pi]$

or sometimes $\theta \in [0, 2\pi)$

occasionally, θ is left open-ended, so no restrictions but the angle within $(-\pi, \pi)$ is called the principle angle

note: in physics and engineering, a phasor (phase vector) is a complex quantity representing a sinusoidal function

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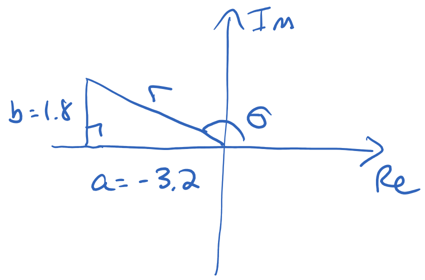
example: convert $z = -3.2 + 1.8i$ to polar notation. Round all values to one decimal place.

answer:



$$r = \sqrt{a^2+b^2}$$
$$= \sqrt{(-3.2)^2 + (1.8)^2}$$
$$\approx 3.67 \approx 3.7$$

$$\tan\theta = \frac{b}{a} = \frac{1.8}{-3.2}$$

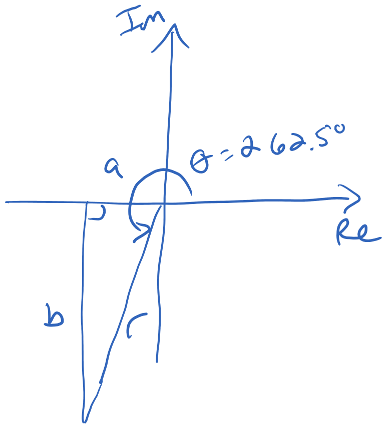
$$\arctan\left(\frac{1.8}{-3.2}\right) \approx -29.36^\circ$$
$$\approx -29.4^\circ$$

to get θ in QII, add 180°

so $\theta \approx 150.6^\circ$

so $z = 3.7\angle 150.6^\circ$

b) convert $z = 2.6\angle 262.5^\circ$ to rectangular form



$a = r\cos\theta$
$= 2.6 \cos 262.5^\circ$
≈ -0.34

$b = r\sin\theta$
$= 2.6 \sin 262.5^\circ$
≈ -2.58

$z = a + bi$
$= -0.34 - 2.58i$

Basic Operations with Complex Numbers:

① addition: easiest with rectangular form

$$(3+5i) + (-1+4i) = 2+9i$$

so if your complex numbers are in polar form, then convert them to rectangular before adding

② multiplication:

rectangular form:

$$(3+5i)(-1+4i) = -3+12i-5i+20i^2$$
$$= -3+7i+20(-1)$$
$$= -23+7i$$

polar form: easiest method is to write your complex numbers in exponential form

$$re^{i\theta}$$

so multiply $3e^{i\pi/3} \cdot 5e^{i\pi/2} = 15e^{i(\pi/3 + \pi/2)}$

$$= 15e^{i5\pi/6}$$

or you could, if you insist, memorize

$$z_1 z_2 = r_1 r_2 \angle \theta_1 + \theta_2$$

③ division:

rectangular form: $\dfrac{z_1}{z_2} = \dfrac{3+5i}{-1+2i}\left(\dfrac{-1-2i}{-1-2i}\right)$

$$= \frac{-3-6i-5i-10i^2}{1-4i^2}$$

$$= \frac{-3 \quad -11i \quad +10}{1+4}$$

$$= \frac{7-11i}{5} \quad \text{or} \quad \frac{7}{5} - \frac{11}{5}i$$

polar form: $\dfrac{z_3}{z_4} = \dfrac{5e^{i\pi/2}}{10e^{i\pi/3}} = \dfrac{1}{2}e^{i(\pi/2 - \pi/3)}$

$$= \tfrac{1}{2}e^{i\pi/6}$$

or $\dfrac{z_3}{z_4} = \dfrac{r_3 \angle \theta_3}{r_4 \angle \theta_4} = \dfrac{r_3}{r_4} \angle \theta_3 - \theta_4$

the most beautiful equation in mathematics:

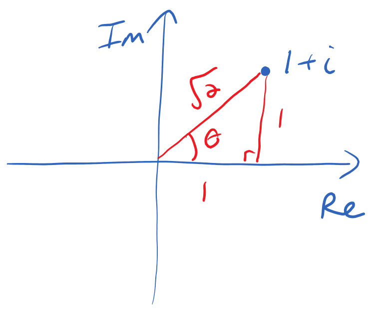
$$e^{i\pi} + 1 = 0$$

Complex Numbers, cont'd

④ Powers of complex numbers

$(1+i)^3$ – if you insist, you could just FOIL this relentlessly, but if the exponent is, say, 9, then this technique becomes far less fun

or



$\theta = 45^\circ$ or $\pi/4$

$z = re^{i\theta}$

$$z^3 = (1+i)^3 = \left(\sqrt{2}\, e^{i\pi/4}\right)^3$$

$$= (\sqrt{2})^3 e^{i3\pi/4}$$

$$= 2\sqrt{2}\, e^{i3\pi/4}$$

or $z^n = r^n \angle n\theta$

Roots of Complex Numbers:

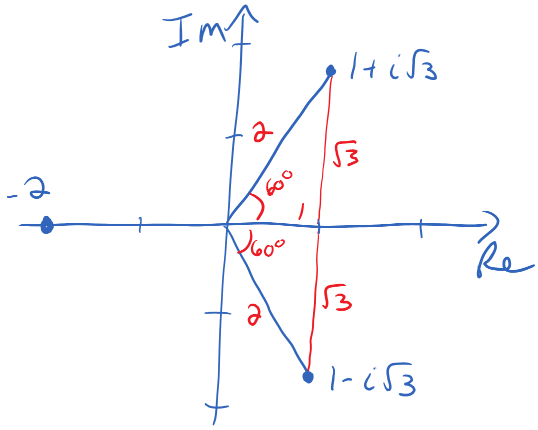
example: $x^3 = -8$ find <u>all</u> solutions

$x^3 = -8$ has three solutions

$x = -2$

$x = 1 - i\sqrt{3}$

$x = 1 + i\sqrt{3}$



how many solutions does $x^4 = 16$ have?

algebraically, we usually just find the real solutions

$$x = \pm 2$$

<u>but</u>

$$x^4 = 16$$
$$x^4 - 16 = 0$$
$$(x^2+4)(x^2-4) = 0$$

$x^2 + 4 = 0$ or $x^2 - 4 = 0$

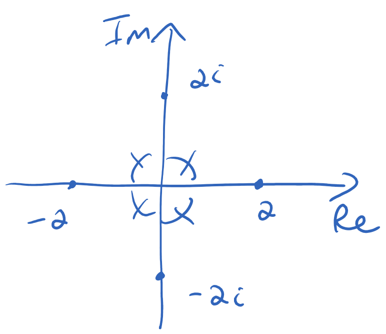
$x^2 = -4$ $\quad$ $x^2 = 4$

$x = \pm 2i$ $\quad$ $x = \pm 2$

$(x-2i)(x+2i)(x-2)(x+2) = 0$

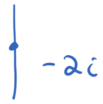
$x = \pm 2, \pm 2i$

a better method: how do you find n^{th} roots in general?



$x^4 = 16$

has four answers, all evenly spaced in the complex plane



plane

the plan: to find all n^{th} roots of a complex number, find one of them algebraically, the rest of them will be evenly spaced
- n in total with angle $\frac{360°}{n}$ between them

in general, $z = re^{i\theta}$

then $\sqrt[n]{z} = z^{1/n} = (re^{i\theta})^{1/n}$

$= r^{1/n} e^{i\theta/n}$

gives you the first root

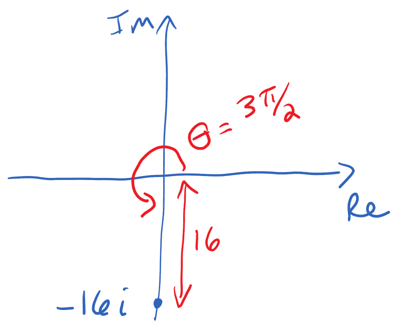
nitpicker note:

DeMoivre's theorem

$$z^{1/n} = r^{1/n} e^{i\frac{1}{n}(\theta + 360°\cdot k)}, \quad k = 0, 1, 2, \ldots n-1$$

example: solve $z^4 + 16i = 0$

answer:



$z^4 = -16i$

$= 16\, e^{i\,3\pi/2}$

$z = (16\, e^{i\,3\pi/2})^{1/4}$

$= 2e^{i\,3\pi/8}$

this is one of the fourth roots - the others are $\frac{360°}{n} = \frac{360°}{4} = 90°$ or $\frac{\pi}{2}$ apart all answers:

$$z = 2e^{i\,3\pi/8},\ 2e^{i(3\pi/8+\pi/2)},\ 2e^{i(3\pi/8+\pi)},\ 2e^{i(3\pi/8+\frac{3\pi}{2})}$$

$$= 2e^{i\,3\pi/8},\ 2e^{i\,7\pi/8},\ 2e^{i\,11\pi/8},\ 2e^{i\,15\pi/8}$$

$\underline{\underline{\text{or}}}$ $\quad z = 2\angle 67.5^\circ,\ 2\angle 157.5^\circ,\ 2\angle 247.5^\circ,\ 2\angle 337.5^\circ$

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note: <u>any</u> polynomial with real coefficients that has degree n has n complex roots.

example: $\quad x^2 + 4x + 13 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= -2 \pm 3i$$