Exploration: The Cross Product

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in $\mathbb{R}^3$, let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

then the cross product of these vectors

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_1 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

DO NOT MEMORIZE

single number

so the result is a vector

here's the shortcut:

recall the unit vectors $\hat{i}, \hat{j}, \hat{k}$

then you could rewrite the vector

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad \text{as} \quad \vec{u} = 2\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

← the absolute value bars mean we are calculating the determinant - more in chapter 4

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \underbrace{\begin{matrix} \hat{i} & \hat{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{matrix}}_{\text{repeat first two columns}}$$

repeat first two colums

take the product along the highlighted lines
add and subtract

example: compute $\vec{u} \times \vec{v}$ for $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

answer: $\vec{u} \times \vec{v} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 1 & 2 & -1 & 1 & 2 \\ 2 & 3 & 1 & 2 & 3 \end{array} \right.$

$$= \hat{i}(2)(1) + \hat{j}(-1)(2) + \hat{k}(1)(3) - \hat{k}(2)(2) - \hat{i}(-1)(3) - \hat{j}(1)(1)$$

$$= 2\hat{i} - 2\hat{j} + 3\hat{k} - 4\hat{k} + 3\hat{i} - \hat{j}$$

$$\left.\begin{aligned} &= 5\hat{i} - 3\hat{j} - \hat{k} \\ &= \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix} \end{aligned}\right\} \text{either}$$

note: the result of the cross product is also a vector, one that is perpendicular to the plane containing the first two

for fun, let's do a couple of computations:

$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\vec{u} \times \vec{v} = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}$

$$\vec{u}\cdot(\vec{u}\times\vec{v}) = \begin{bmatrix}1\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}5\\-3\\-1\end{bmatrix} = 5-6+1 = 0$$

$$\vec{v}\cdot(\vec{u}\times\vec{v}) = \begin{bmatrix}2\\3\\1\end{bmatrix}\cdot\begin{bmatrix}5\\-3\\-1\end{bmatrix} = 10-9-1 = 0$$

so $(\vec{u}\times\vec{v})$ is perpendicular to $\vec{u}$ <u>and</u>
perpendicular to $\vec{v}$

the direction of $\vec{u}\times\vec{v}$ can be found using the right-hand rule

- make your right hand flat with the thumb at a right angle
- stick your thumb along the direction of the first vector
- fingers go along direction of second vector

then palm pushes in direction of cross product

$\overrightarrow{\text{first}} \times \overrightarrow{\text{second}}$

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from the right-hand rule, we see the order of the vectors is important

and

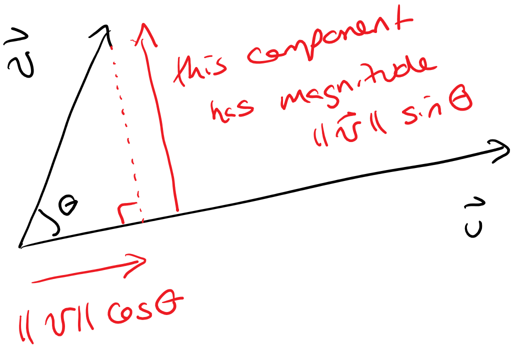
$$\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$$

here's the property of the cross product that you really need to know:

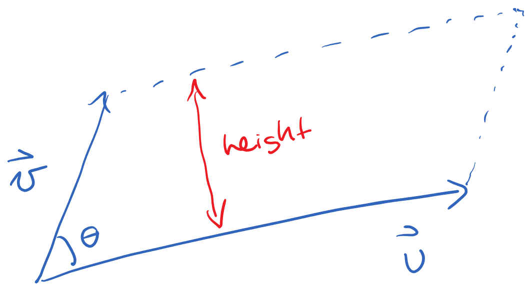
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$$

where θ = angle between $\vec{u}$ and $\vec{v}$

(recall: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$)



there's a nice physical representation:



make a parallelogram from $\vec{u}$ and $\vec{v}$

if $\vec{u}$ is base, then the "height" is $\|\vec{v}\| \sin\theta$

so area of parallelogram $= \|\vec{u}\| \, \|\vec{v}\| \sin\theta$

example: what is the area of triangle ABC with vertices $A = (1, 0, 1)$, $B = (0, 2, 3)$, and $C = (2, 1, 0)$?

answer:

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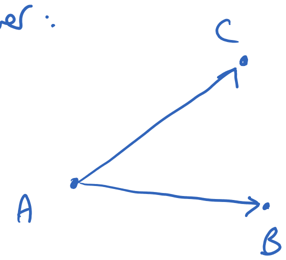


$$\vec{AB} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{AC} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{AB} \times \vec{AC} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -1 & 2 & 2 & -1 & 2 \\ 1 & 1 & -1 & 1 & 1 \end{array} \right|$$

$$= -2\hat{i} + 2\hat{j} - \hat{k} - 2\hat{k} - 2\hat{i} - \hat{j}$$

$$\left.\begin{aligned} &= -4\hat{i} + \hat{j} - 3\hat{k} \\ &= \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix} \end{aligned}\right\} \text{either is fine}$$

$$\text{area of } \triangle ABC = \tfrac{1}{2} \| \vec{AB} \times \vec{AC} \|$$

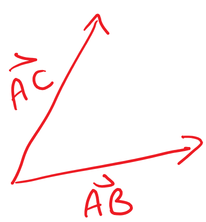
$$= \tfrac{1}{2} \sqrt{(-4)^2 + (1)^2 + (-3)^2}$$

$$= \tfrac{1}{2} \sqrt{26}$$

example: Find vector $\vec{N}$ $\perp$ to the plane containing point ABC where

$A = (-4, 0, 2)$, $B = (1, -3, 1)$, and $C = (2, -2, 6)$

answer:



$$\vec{AB} = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix} \qquad \vec{AC} = \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$$

vector $\vec{N}$ is

perpendicular to plane containing $\vec{AB}$ and $\vec{AC}$

$$\vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -3 & -1 \\ 6 & -2 & 4 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 5 & -3 \\ 6 & -2 \end{matrix}$$

$$= -12\hat{i} - 6\hat{j} - 10\hat{k} + 18\hat{k} - 2\hat{i} - 20\hat{j}$$

$$\vec{N} = -14\hat{i} - 26\hat{j} + 8\hat{k}$$

note: if you did $\vec{AC} \times \vec{AB}$, then $14\hat{i} + 26\hat{j} - 8\hat{k}$

is another perfectly acceptable answer

example: Find a unit vector parallel to the yz-plane that is perpendicular to $\vec{\omega}$, where

$$\vec{\omega} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

method (1): if $\vec{u}$ is the vector we want, then $\vec{u} \cdot \vec{\omega} = 0$

and if $\vec{u}$ is parallel to the yz-plane,

then $\vec{u} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$ ⇐ no x-component

then $\vec{u} \cdot \vec{\omega} = 0$

$$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 0$$

$$0 - a + 2b = 0$$

$$a = 2b$$

but $\vec{u}$ is a unit vector, so

$$\sqrt{0^2 + a^2 + b^2} = 1$$

$$\sqrt{4b^2 + b^2} = 1$$

$$5b^2 = 1$$

$$b = \pm \frac{1}{\sqrt{5}}$$

be careful! don't say that ~~$b = \pm \frac{1}{\sqrt{5}}$ and $a = \pm \frac{1}{\sqrt{5}}$~~

say if $b = \frac{1}{\sqrt{5}}$, then $a = \frac{2}{\sqrt{5}}$

$b = -\frac{1}{\sqrt{5}}$ $\quad a = -\frac{2}{\sqrt{5}}$

then $\vec{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

note: <u>all</u> unit vectors: $\vec{u} = \pm \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

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method #2: let the vector of interest be $\vec{u}$

since $\vec{u}$ is parallel to the yz-plane, it is perpendicular to the x-axis

so $\vec{u} \perp$ x axis

$\vec{u} \perp \hat{i}$

and from question, $\vec{u} \perp \vec{w}$

so $\vec{u} \parallel \vec{w} \times \hat{i}$

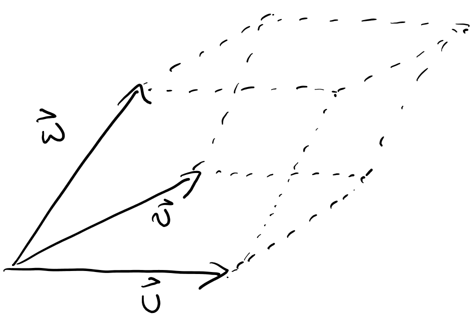
$$\vec{w} \times \hat{i} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 3 & -1 & 2 & 3 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right.$$

$$= 2\hat{j} + \hat{k}$$

$$\|\vec{w} \times \hat{i}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

so $\vec{u} = \dfrac{\vec{w} \times \hat{i}}{\|\vec{w} \times \hat{i}\|} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

one last thing:



volume $= |\vec{w} \cdot (\vec{u} \times \vec{v})|$