Date: Fall 2023

Instructor: Patricia Wrean

Name: Solution Set

Math 251
Test 1

Total = $\overline{}$ 25

Show your work. All of the work on this test must be your own.

GOOD LUCK!

1. (4 points) Consider the following vectors.

$$\mathbf{u} = \begin{bmatrix} x \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$$

Find all values of x for which:

(a) **u** and **v** are perpendicular.

$\vec{u} \cdot \vec{v} = 0$

1

$4x - 2 - 18 = 0$

$4x = 20$

$\boxed{x = 5}$

(b) **u** and **v** are parallel.

$\vec{v} = k\vec{u}$

$\begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} = k \begin{bmatrix} x \\ -1 \\ 3 \end{bmatrix}$ So $k = -2$

1

and $\boxed{x = -2}$

(c) **u** and **v** have the same norm.

$\| \vec{u} \| = \| \vec{v} \|$

$\sqrt{x^2 + 1 + 9} = \sqrt{16 + 4 + 36}$

2

$\sqrt{x^2 + 10} = \sqrt{56}$

$x^2 + 10 = 56$

$\boxed{\begin{aligned} x &= \pm\sqrt{46} \\ &\approx \pm 6.78 \end{aligned}}$

2. (5 points) Consider the following three points.

$$A = (0, 1, -2), \quad B = (1, 2, 1), \quad C = (1, 4, -3)$$

(a) Find the angle $0 \leq \theta \leq 180^\circ$ between vectors **AB** and **BC**.

$\vec{AB} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{BC} = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$

$\vec{AB} \cdot \vec{BC} = \|\vec{AB}\| \, \|\vec{BC}\| \cos\theta$

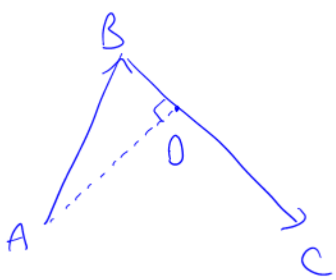
2

$\cos\theta = \dfrac{\vec{AB} \cdot \vec{BC}}{\|\vec{AB}\| \, \|\vec{BC}\|}$

$= \dfrac{0 + 2 - 12}{\sqrt{1+1+9} \, \sqrt{0+4+16}} = \dfrac{-10}{\sqrt{11}\sqrt{20}}$

$\boxed{\theta \approx 132.4^\circ \text{ or } 2.31 \text{ rads}}$

(b) Find the vector component of **AB** along **BC**.



$\vec{DB} = \text{proj}_{\vec{BC}}(\vec{AB}) = \dfrac{\vec{BC} \cdot \vec{AB}}{\vec{BC} \cdot \vec{BC}} \vec{BC}$

$= \dfrac{-10}{20} \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$

2

$= \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

(c) Find the vector component of **AB** perpendicular to **BC**.

$\vec{AB} = \vec{AD} + \vec{DB}$

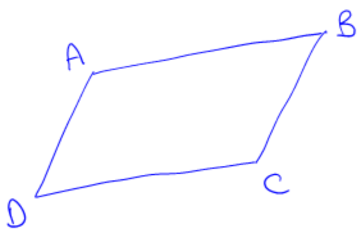
so $\vec{AD} = \vec{AB} - \vec{DB}$

1

$= \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

3. (5 points) The parallelogram $ABCD$ has vertices at $A = (1, 1, -1)$, $B = (-3, 2, -2)$, and $C = (-2, 2, 1)$. The convention is that the points are named in order as you go around the perimeter of the geometric figure. So point A is connected to points B and D.

(a) Is this parallelogram a rectangle? Explain briefly.



$\vec{AB} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$ $\quad \vec{BC} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

if $\vec{AB} \perp \vec{BC}$, then the dot product equals zero

$\vec{AB} \cdot \vec{BC} = -4(1) + 0 + (-1)(3) \neq 0$

2

$\boxed{\text{No, not a rectangle}}$

(b) Is this parallelogram a rhombus? (A rhombus is a parallelogram with all four sides of equal length.) Explain briefly.

$\|\vec{AB}\| = \sqrt{16 + 1 + 1} = \sqrt{18}$

$\|\vec{BC}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$

2

sides are of different length, so

$\boxed{\text{No, not a rhombus}}$

(c) Calculate the coordinates of point D.

$\vec{AD} = \vec{D} - \vec{A}$ and $\vec{AD} = \vec{BC}$

1

so $\vec{D} = \vec{A} + \vec{AD}$

$= A + \vec{BC}$

$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$\boxed{\text{point } D = (2, 1, 2)}$

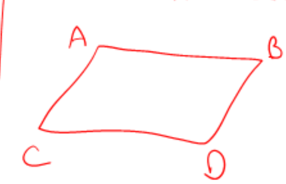
or $\vec{AB} = \vec{DC}$

and $\vec{DC} = \vec{C} - \vec{D}$

$\vec{D} = \vec{C} - \vec{DC} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$

for same result

note: if used



get $(-6, 3, 0)$

$-\frac{1}{2}$

4. (5 points) Consider the following plane.

$$\begin{cases} x &= 5 - s + t \\ y &= 4 + 2s - t \\ z &= 1 - 2s \end{cases}$$

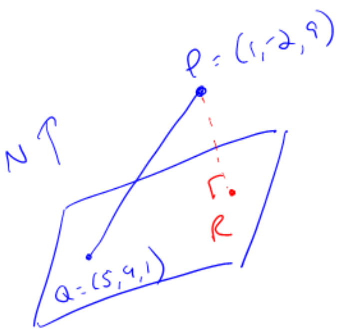
Calculate the distance from this plane to the point $P = (1, -2, 9)$.

the plane contains the point $(5, 4, 1)$ and vectors $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ 1

normal is then $\vec{u} \times \vec{v} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -1 & 2 & -2 & -1 & 2 \\ 1 & -1 & 0 & 1 & -1 \end{array} \right.$

$= -2\hat{j} + \hat{k} - 2\hat{k} - 2\hat{i} = -2\hat{i} - 2\hat{j} - \hat{k}$

$\vec{N} = \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}$ 1



$\vec{PQ} = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix}$ 1

$\vec{PR} = \text{proj}_{\vec{N}}(\vec{PQ})$

$= \dfrac{\vec{N} \cdot \vec{PQ}}{\vec{N} \cdot \vec{N}} \vec{N}$

$= \dfrac{-8 - 12 + 8}{4 + 4 + 1} \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} = \dfrac{-12}{9} \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} = \dfrac{4}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ 1

distance $= \|\vec{PR}\| = \dfrac{4}{3}\sqrt{4 + 4 + 1} = \dfrac{4}{3} \cdot 3 = 4$

$\boxed{\text{distance} = 4}$ 1

5. (6 points) Consider the following systems.

(a) Use Gauss-Jordan elimination to solve the following system and be sure to specify which row operations you are using. Write your answer in column vector form.

$$\begin{cases} 3x + y + 5z = -11 \\ x - y + 3z = -9 \\ -2x + y - 5z = 14 \end{cases}$$

RREF 3

solution 1

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & -11 \\ 1 & -1 & 3 & -9 \\ -2 & 1 & -5 & 14 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -9 \\ 3 & 1 & 5 & -11 \\ -2 & 1 & -5 & 14 \end{array}\right]$$

only −1 row-echelon, no RREF

$$\begin{array}{c} R_2 - 3R_1 \\ R_3 + 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -9 \\ 0 & 4 & -4 & 16 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ \tfrac{1}{4} R_2 \\ R_3 + \tfrac{1}{4} R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 2z = -5$
$y - z = 4$

let $z = t$

then $\begin{cases} x = -2t - 5 \\ y = t + 4 \\ z = t \end{cases}$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}}$$

(b) For which values of h and k does the following system have no solutions? (Use any method.)

$$\begin{cases} x + 6y = h \\ -4x + ky = 10 \end{cases}$$

mult by −4

$$\begin{cases} -4x - 24y = -4h \\ -4x + ky = 10 \end{cases}$$

2

so $k = -24$

$-4h \neq 10$

$$\boxed{\begin{array}{c} k = -24 \\ h \neq -\tfrac{5}{2} \end{array}}$$