Math 251 – Test 1

October 5, 2018 **Name:** Solution Set

Instructor: Patricia Wrean

Total: 25 points

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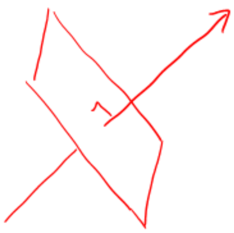
1. (4 points) Give an equation in general form for the plane passing through the point $P = (3, -2, 1)$ and perpendicular to the line with the following parametric equations.

$$x = 4 - 2t, \quad y = -2 + t, \quad z = 3 - 5t$$



$x = 4 - 2t$
$y = -2 + t$
$z = 3 - 5t$

direction vector

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$ 1

if the plane is perpendicular to the line, it is perp to $\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ 1

so the normal to the plane is parallel to $\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ and 1

$$ax + by + cz = d$$
$$-2x + y - 5z = d$$

now sub in point P

$$-2(3) + (-2) - 5(1) = d$$
$$d = -13$$

so $\boxed{\begin{array}{l} -2x + y - 5z = -13 \\ \text{or } 2x - y + 5z = 13 \end{array}}$ 1

2. (6 points) Consider the following three points.

$$A = (0, 1, -2), \quad B = (1, 2, 1), \quad C = (-1, 2, -3)$$

(a) Give a unit vector that is parallel to $\overrightarrow{AC}$.

(b) Calculate the angle $0 \le \theta \le 180°$ between vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(c) Calculate the vector component of $\overrightarrow{AC}$ along $\overrightarrow{BC}$.

$$\vec{AB} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad \vec{AC} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \qquad \vec{BC} = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$$

a) $\hat{u} = \pm \dfrac{\vec{AC}}{\|\vec{AC}\|} = \pm \dfrac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ 1

↑ since the question asked for a vector, not all vectors, can omit $\pm$

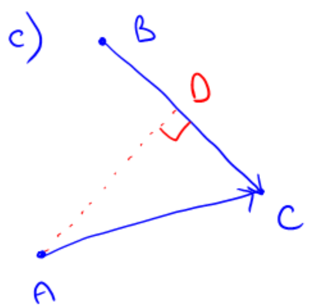
b) $\cos\theta = \dfrac{\vec{AB} \cdot \vec{BC}}{\|\vec{AB}\| \|\vec{BC}\|}$ where $\vec{AB} \cdot \vec{BC} = -2 - 12 = -14$

$= \dfrac{-14}{\sqrt{11}\sqrt{20}}$ $\quad \|\vec{AB}\| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$

$\|\vec{BC}\| = \sqrt{(-2)^2 + 0 + (-4)^2} = \sqrt{20}$ 3

$$\boxed{\begin{aligned} \theta &\approx 160.7° \\ &\approx 2.81 \text{ rads} \end{aligned}}$$

c)

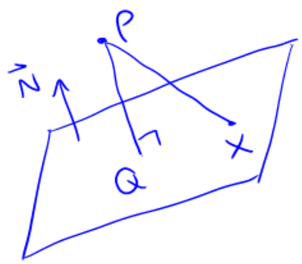


$\vec{DC} = \text{proj}_{\vec{BC}}(\vec{AC}) = \dfrac{\vec{BC} \cdot \vec{AC}}{\vec{BC} \cdot \vec{BC}} \vec{BC}$

$= \dfrac{2 + 4}{4 + 16} \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$ 2

$= \dfrac{3}{10} \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} = \boxed{-\dfrac{3}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$

3. (5 points) Calculate the distance from the point $P = (1, 0, 1)$ to the plane $x - 2y + 2z = 1$.



the point $X = (1,0,0)$ is on the plane

$$\vec{N} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \qquad \vec{XP} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1

$$\vec{QP} = \text{proj}_{\vec{N}} (\vec{XP})$$

$$= \frac{N \cdot \vec{XP}}{\vec{N} \cdot \vec{N}} \vec{N}$$

$$= \frac{2}{1+4+4} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

3

$$= \frac{2}{9} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

distance $d = \| \vec{QP} \| = \frac{2}{9} \sqrt{1 + 4 + 4}$

$\boxed{= 2/3}$

1

4. (4 points) Consider the line

$$\begin{cases} x - 2y + 2z = 3 \\ 2x - 5y + 3z = 1 \end{cases}$$

$\vec{N}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ $\vec{N}_2 = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$

Is the vector $\mathbf{v} = \begin{bmatrix} -8 \\ -2 \\ 2 \end{bmatrix}$ parallel to this line? Explain your reasoning.

method #1: the line is $\perp$ to both normals

$$\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 2 & -5 & 3 \end{vmatrix} = -6\hat{i} + 4\hat{j} - 5\hat{k} + 4\hat{k} + 10\hat{i} - 3\hat{j} \quad 2$$

$$= 4\hat{i} + \hat{j} - \hat{k}$$

so line is parallel to $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ 1

and since $\vec{v} = \begin{bmatrix} -8 \\ -2 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$, then $\boxed{\text{yes}}$ 1

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method #2:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 2 & -5 & 3 & 1 \end{array}\right] \xrightarrow[R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & -1 & -1 & -5 \end{array}\right] \xrightarrow[-R_2]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_1 + 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 13 \\ 0 & 1 & 1 & 5 \end{array}\right] \quad 1$$

so let $z = t$

$$\begin{cases} x = 13 - 4t \\ y = 5 - t \\ z = t \end{cases} \quad 1$$

and direction vector is $\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$ 1

and since $v = \begin{bmatrix} -8 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$ then $\boxed{\text{yes}}$ 1

5. (6 points) Consider the system of equations below.

$$\begin{cases} x & -2y & +3z & = & 2 \\ x & +2y & +z & = & 0 \\ 2x & & +4z & = & m \end{cases}$$

For what values of m, if any, does the system have

(a) no solutions

(b) one solution

(c) infinitely many solutions?

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 4 & m \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & 4 & -2 & m-4 \end{array}\right]$$

$$\downarrow$$

$$R_3-R_2 \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & 0 & 0 & m-2 \end{array}\right]$$

3

if $m=2$, get

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinitely many solutions

if $m \neq 2$, get

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & 0 & 0 & m-2 \end{array}\right] \leftarrow \text{a non-zero number}$$

so no solutions

so

$$\boxed{\begin{array}{l} \text{a) } m \neq 2 \\ \text{b) none} \\ \text{c) } m = 2 \end{array}}$$

1
1
1