Math 251 – Test 2

November 2, 2018 **Name:** Solution Set
Instructor: Patricia Wrean

Total: 25 points

1. (5 points) Find the 2×2 matrix A that satisfies the following.

$$BA - C^T = 3B, \text{ where } B = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$$

$$BA = 3B + C^T$$

$$B^{-1}BA = B^{-1}(3B + C^T)$$

$$IA = 3B^{-1}B + B^{-1}C^T$$

$$A = 3I + B^{-1}C^T$$

where $B^{-1} = \frac{1}{-6+5}\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$

and $C^T = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$

$$A = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 8 & 11 \end{bmatrix}$$

2. (5 points) Solve the system $A\mathbf{x} = \mathbf{b}$ using the following factorization of A.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}, \mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 14 \\ -8 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

$L\underbrace{U\vec{x}} = \vec{b}$

let $\vec{y} = U\vec{x}$ then $L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 14 \\ -8 \end{bmatrix}$$

so $y_1 = 5$

$y_1 + y_2 = 6$ and $y_2 = 1$

$2y_1 + y_3 = 14$ and $y_3 = 4$

$-y_1 - y_2 + y_4 = -8$ and $y_4 = -2$

but $\vec{y} = U\vec{x}$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \\ -2 \end{bmatrix}$$

$x_1 - 2x_2 + 3x_3 + x_4 = 5$ and $x_1 = -25$

$x_2 + 2x_3 + 2x_4 = 1$ and $x_2 = -7$

$x_3 + x_4 = 4$ and $x_3 = 6$

so $x_4 = -2$

giving

$$\vec{x} = \begin{bmatrix} -25 \\ -7 \\ 6 \\ -2 \end{bmatrix}$$

3. (5 points) Consider the following matrix.

$$A = \begin{bmatrix} 2 & -2 & 4 \\ -5 & 5 & -10 \\ 2 & 1 & -5 \\ 4 & -7 & 17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the row space of A consisting of rows of A.

(b) Find a basis for the column space of A consisting of columns of A.

a) (1) $A^T = \begin{bmatrix} 2 & -5 & 2 & 4 \\ -2 & 5 & 1 & -7 \\ 4 & -10 & -5 & 17 \end{bmatrix} \xrightarrow[R_3+2R_2]{R_2+R_1} \begin{bmatrix} 2 & -5 & 2 & 4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \xrightarrow[R_3+R_2]{\frac{1}{3}R_2} \begin{bmatrix} 2 & -5 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1-2R_2} \begin{bmatrix} 2 & -5 & 0 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -5/2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (1)

(4)

so a basis for col (A^T) is

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}$$ (1)

so a basis for row $(A) = \{ [2 \ -2 \ 4], [2 \ 1 \ -5] \}$ (1)

b) basis for col (A):

$$\left\{ \begin{bmatrix} 2 \\ -5 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \\ -7 \end{bmatrix} \right\}$$ (1)

(columns corresponding to leading ones in original RREF)

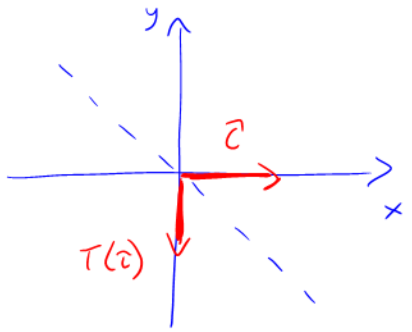
4. (5 points) Find the standard matrix A for the following transformations. Note that

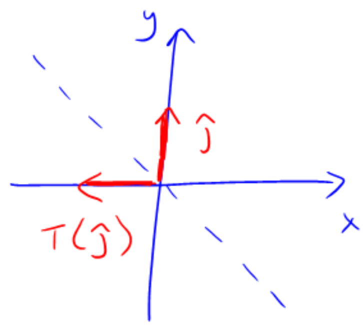
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(a) The transformation which reflects a vector in $\mathbb{R}^2$ about the line $y = -x$.

(b) The transformation which reflects a vector in $\mathbb{R}^2$ about the line $y = -x$ and then rotates it counterclockwise by 30°.

a)





$T(\hat{i}) = -\hat{j} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\qquad$ $T(\hat{j}) = -\hat{i} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

so $A_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

b) $(S \circ T)(\vec{x}) = R_\theta A_1 (\vec{x})$

$$A_2 = R_\theta A_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

where $\theta = 30°$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

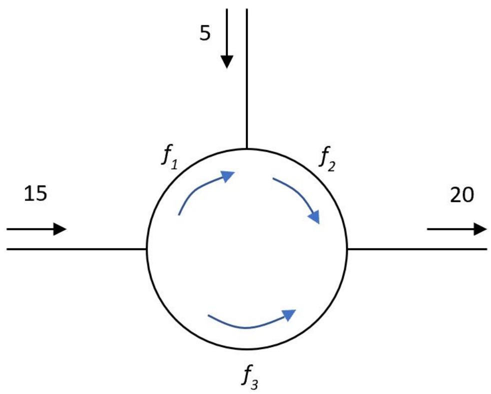
$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

5. (5 points) City engineers have been measuring the traffic flow for the streets surrounding a circular park, as shown in the diagram below. Each street is one-way, and the numbers represent the average number of vehicles per minute entering and leaving the intersections during the day.

   (a) Set up and solve a system of linear equations to find the possible flows f_1, f_2, and f_3.

   (b) What are the minimum and maximum possible flows on each street?

a)



$$\begin{cases} 15 = f_1 + f_3 \\ f_1 + 5 = f_2 \\ f_2 + f_3 = 20 \end{cases}$$

so

$$\begin{cases} f_1 + f_3 = 15 \\ f_1 - f_2 = -5 \\ f_2 + f_3 = 20 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 1 & -1 & 0 & -5 \\ 0 & 1 & 1 & 20 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 0 & -1 & -1 & -20 \\ 0 & 1 & 1 & 20 \end{array}\right] \xrightarrow[R_3 + R_2]{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so let $f_3 = t$

then

$$\begin{cases} f_1 = 15 - t \\ f_2 = 20 - t \\ f_3 = t \end{cases}$$

b) all traffic flows must be non-negative

so $\begin{cases} f_1 \geq 0 \\ f_2 \geq 0 \\ f_3 \geq 0 \end{cases}$ and $\begin{cases} 15 - t \geq 0 \\ 20 - t \geq 0 \\ t \geq 0 \end{cases}$ which means that $t \leq 15$, $t \leq 20$, $t \geq 0$

we conclude that $0 \leq t \leq 15$

and

$$\boxed{\begin{array}{c} 0 \leq f_1 \leq 15 \\ 5 \leq f_2 \leq 20 \\ 0 \leq f_3 \leq 15 \end{array}}$$