Math 251 – Test 3

November 30, 2018 **Name:** Solution Set
Instructor: Patricia Wrean

Show your work to receive full credit. **Total: 25 points**

1. (5 points) Consider the following complex numbers.

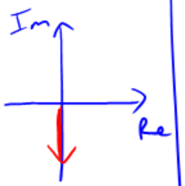
$$z_1 = -2i, z_2 = 3 - 4i, z_3 = 4e^{i\pi/4}$$

Evaluate the following. You may leave your answer in either rectangular or polar form, your choice. Give an exact answer for part (a).

(a) $z_1 + z_3$

(b) $\frac{z_1}{z_2}$

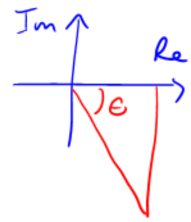
$z_1 = -2i$



$= 2e^{i\,3\pi/2}$ or $2e^{-i\pi/2}$

$z_2 = 3-4i$

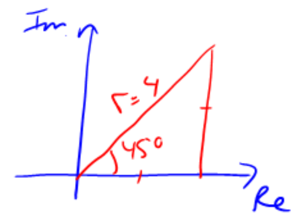
$= re^{i\theta}$ where



$r = \sqrt{a^2+b^2} = 5$

$\tan^{-1}(-4/3) = -53.1°$

$= -0.927$ rads



$z_3 = 4e^{i\pi/4}$

$= 2\sqrt{2} + 2\sqrt{2}\,i$

a) $z_1 + z_3 = -2i + 2\sqrt{2} + 2\sqrt{2}\,i$

$= 2\sqrt{2} + i(-2+2\sqrt{2})$

(-1/2) for $2.83 + 0.83i$

b) method #1:

$$\frac{z_1}{z_2} = \frac{-2i}{3-4i}\left(\frac{3+4i}{3+4i}\right)$$

$$= -\frac{6i+8}{9+16} = \boxed{\frac{8}{25} - \frac{6}{25}i}$$

(1) correct conversion

(1) each of r, θ

method #2:

$$\frac{z_1}{z_2} = \frac{2e^{i\,3\pi/2}}{5e^{-0.927i}} \approx \boxed{\frac{2}{5}e^{5.64i}} \text{ or } \frac{2}{5}e^{-0.643i}$$

2. (5 points) Use Cramer's Rule to solve the following system of linear equations.

$$\begin{cases} x + 2y - z = 2 \\ 3x + 7y - 5z = 5 \\ -x - 2y = 1 \end{cases}$$

solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

(-1) each incorrect matrix

(-½) arithmetic error (if enough detail I can see it)

(-1) major error / not enough detail that I can tell if it's arithmetic or not

and $\det(A) = 10 + 6 - 7 - 10 = -1$

$A_1(\vec{b}) = \begin{bmatrix} 2 & 2 & -1 \\ 5 & 7 & -5 \\ 1 & -2 & 0 \end{bmatrix}$ $\det(A_1(\vec{b})) = 0 - 10 + 10 + 7 - 20 - 0 = -13$

$A_2(\vec{b}) = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -5 \\ -1 & 1 & 0 \end{bmatrix}$ $\det(A_2(\vec{b})) = 10 - 3 - 5 + 5 = 7$

$A_3(\vec{b}) = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 5 \\ -1 & -2 & 1 \end{bmatrix}$ $\det(A_3(\vec{b})) = 7 - 10 - 12 + 14 + 10 - 6 = 3$

So $x = \dfrac{\det(A_1(\vec{b}))}{\det(A)} = \dfrac{-13}{-1} = 13$

$y = \dfrac{7}{-1} = -7$

$z = \dfrac{3}{-1} = -3$

$\boxed{(x, y, z) = (13, -7, -3)}$

3. (3 points) Find all values of x for which the following matrix is not invertible.

$$A = \begin{bmatrix} x & x & 0 \\ x & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix} \begin{matrix} x & x \\ x & 3 \\ 2 & 4 \end{matrix}$$

$\det(A) = 0$

$9x + 2x + 0 - 0 - 4x - 3x^2 = 0$

$7x - 3x^2 = 0$

$x(7 - 3x) = 0$

$\boxed{x = 0, \ 7/3}$

(1) correct method

(1) $x = 0$ by any method

(1) $x = 7/3$

See next page for alternate approach

(3)

$$A = \begin{bmatrix} x & x & 0 \\ x & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

A is not invertible if RREF does not give I

$$\begin{bmatrix} x & x & 0 \\ x & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix} \xrightarrow[xR_3]{R_2 - R_1} \begin{bmatrix} x & x & 0 \\ 0 & 3-x & 1 \\ 2x & 4x & 3x \end{bmatrix} \xrightarrow[R_3 - 2R_1]{} \begin{bmatrix} x & x & 0 \\ 0 & 3-x & 1 \\ 0 & 2x & 3x \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{r} \\ 2xR_2 \\ (3-x)R_3 \end{array} \begin{bmatrix} x & x & 0 \\ 0 & 2x(3-x) & 2x \\ 0 & 2x(3-x) & 3x(3-x) \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} x & x & 0 \\ 0 & 2x(3-x) & 2x \\ 0 & 0 & 2x - 3x(3-x) \end{bmatrix}$$

matrix is not invertible if this entry is zero

$$2x - 3x(3-x) = 0$$

$$2x - 9x + 3x^2 = 0$$

$$3x^2 - 7x = 0$$

$$x(3x-7) = 0$$

$$x = 0, \ 7/3$$

4. (6 points) Consider the following matrix.

$$A = \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

find eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 2 \\ -3 & 5-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(5-\lambda) + 6 = 0$$

$$-10 - 3\lambda + \lambda^2 + 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = -1, 4$$

eigenvectors: solve $(A - \lambda I)\vec{x} = 0$

$\lambda_1 = -1$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right] \xrightarrow[R_2 - 3R_1]{-R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free var

let $y = t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 4$

$$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - \frac{1}{2}R_1]{-\frac{1}{6}R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

then $P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

5. (6 points) Consider $\mathbf{v}$ and subspace W.

$$\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \qquad W = \operatorname{span}\left(\overset{\vec{x}_1}{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}, \overset{\vec{x}_2}{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}} \right)$$

(a) Find an orthogonal basis for W.

(b) For the basis you found in part (a), express $\mathbf{v}$ as a linear combination of the vectors in that basis.

a) Gram-Schmidt: let the orthogonal basis be $\{\vec{w}_1, \vec{w}_2\}$

then $\vec{w}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

3

and $\vec{w}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{w}_1}(\vec{x}_2)$

$$= \vec{x}_2 - \frac{\vec{w}_1 \cdot \vec{x}_2}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1$$

$$= \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \frac{-2}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10/3 \\ 2/3 \\ -4/3 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix} \text{ or } \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$$

so orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \right\}$

b) $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = c_1 \vec{w}_1 + c_2 \vec{w}_2$ where

$$c_1 = \frac{\vec{w}_1 \cdot \vec{v}}{\vec{w}_1 \cdot \vec{w}_1} = \frac{-1}{6}$$

3

$$c_2 = \frac{\vec{w}_2 \cdot \vec{v}}{\vec{w}_2 \cdot \vec{w}_2} = \frac{25}{30} = 5/6$$

note: if basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix} \right\}, \text{ will get } \vec{v} = -\tfrac{1}{6}\vec{w}_1 + \tfrac{5}{12}\vec{w}_2$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 10/3 \\ 2/3 \\ -4/3 \end{bmatrix} \right\}, \text{ will get } \vec{v} = \frac{-1}{6}\vec{w}_1 + \frac{5}{4}\vec{w}_2$$