Date: November 4, 2020

Instructor: Patricia Wrean

Name: Solution Set

Math 251
Test 2

Total = $\overline{}$ 25

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math251 or the Math 251 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Poole), or any of the texts listed on the Textbook page at http://wrean.ca/math251/math251_textbook.htm
  - the Math 251 D2L website
  - the Math 251 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - the RREF calculator at https://adrianstoll.com/linear-algebra/row-reduction.html
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (4 points) Consider the matrices $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$.

   Evaluate the following, if they exist. If they do not exist, say so.

   (a) AC
   (b) CA
   (c) $C^T + C$
   (d) C^{-1}

a) AC does not exist 1

b) $CA = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 7 \\ 24 & 18 & 21 \end{bmatrix}$ 1

c) $C^T + C = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 6 \end{bmatrix}$ 1

d) $\det C = 2 \cdot 3 - 6 \cdot 1 = 0$ so C^{-1} is undefined 1

2. (2 points) Solve the given matrix equation for X and simplify. Assume that all matrices are invertible.

$$AX = B^T A + A$$

$$A^{-1}AX = A^{-1}B^T A + A^{-1}A$$

$$\underline{I}X = A^{-1}B^T A + I$$

$$\boxed{X = A^{-1}B^T A + I}$$

note: $A^{-1}B^T A \neq B^T A A^{-1}$

-1 because order matters when multiplying matrices

3. (4 points) Find an LU factorization of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$R_3 - \tfrac{1}{5}R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 6/5 \end{bmatrix}$$

2

so $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/5 & 1 \end{bmatrix}$ 2

$$\boxed{\text{so } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 6/5 \end{bmatrix}}$$

4. (4 points) Consider the following 3×4 matrix.

$$A = \begin{bmatrix} 1 & -1 & 5 & -9 \\ 3 & 1 & 3 & -11 \\ -2 & 3 & -13 & 22 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$A^T = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 1 & 3 \\ 5 & 3 & -13 \\ -9 & -11 & 22 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -\frac{11}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Give the values of rank(A) and nullity(A).

½ rank$(A) = 2$ (= number of leading ones in RREF of A)

½ null$(A) = 2$ (= number of free columns in RREF of A)

(b) Find a basis for null(A).

$A\vec{x} = 0$

$$\left[\begin{array}{cccc|c} 1 & -1 & 5 & -9 & 0 \\ 3 & 1 & 3 & -11 & 0 \\ -2 & 3 & -13 & 22 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -5 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & \underbrace{x_3 \quad x_4}_{\text{free variables}} \end{array} \end{array}$$

1

so
$$x_1 + 2x_3 - 5x_4 = 0$$
$$x_2 - 3x_3 + 4x_4 = 0$$

and let $x_3 = s$, $x_4 = t$

1

then
$$\begin{cases} x_1 = -2s + 5t \\ x_2 = 3s - 4t \\ x_3 = s \\ x_4 = t \end{cases}$$

and

1

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ forms a basis for null}(A)}$$

5. (5 points) Consider the set of all parabolas $y = ax^2 + bx + c$ that pass through the following two points.

$$(1, 12), (2, -18)$$

Set up and solve a system to find the possible values of a, b, and c. Give your answer in parametric form.

$$y = ax^2 + bx + c$$

point $(1,12)$: $12 = a \cdot 1^2 + b \cdot 1 + c$ 1

$12 = a + b + c$

point $(2,-18)$: $-18 = a \cdot 2^2 + b \cdot 2 + c$ 1

$-18 = 4a + 2b + c$

system is $\begin{cases} 12 = a + b + c \\ -18 = 4a + 2b + c \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 4 & 2 & 1 & -18 \end{array}\right] \xrightarrow{RREF} \begin{array}{cccc} & & 1 & \\ \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -21 \\ 0 & 1 & \frac{3}{2} & 33 \end{array}\right] & & & \\ a \quad b \quad c & & & \end{array}$$

free variable

so $a - \frac{1}{2}c = -21$

$b + \frac{3}{2}c = 33$ 1

and let $c = t$

$$\boxed{\begin{cases} a = \frac{1}{2}t - 21 \\ b = -\frac{3}{2}t + 33 \\ c = t \end{cases}}$$ 1

6. (6 points) Consider the following four vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix} -10 \\ -9 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

where the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ gives

$$T(\mathbf{v}_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T(\mathbf{v}_3) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

(a) Express $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -10 \\ 2 & -1 & 2 & -9 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -4 \end{array}\right] \quad 1$$

so $\boxed{\vec{u} = 2\vec{v}_1 + 5\vec{v}_2 - 4\vec{v}_3}$ 1

(b) Are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly dependent or linearly independent? Explain briefly.

1

they are LI, since in the matrices above you can see that $[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3] \xrightarrow{RREF} I_3$. 1

(c) Find $T(\mathbf{u})$ for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$.

$$\vec{u} = 2\vec{v}_1 + 5\vec{v}_2 - 4\vec{v}_3$$

so $T(\vec{u}) = 2T(\vec{v}_1) + 5T(\vec{v}_2) - 4T(\vec{v}_3)$ 1

$$= 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ -5 \end{bmatrix}$$ 1