Date: December 2, 2020

Instructor: Patricia Wrean

Name: Solution Set

Math 251
Test 3

Total = $\overline{}$ / 25

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math251 or the Math 251 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Poole), or any of the texts listed on the Textbook page at http://wrean.ca/math251/math251_textbook.htm
  - the Math 251 D2L website
  - the Math 251 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - the RREF calculator at https://adrianstoll.com/linear-algebra/row-reduction.html
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (3 points) The determinant of matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is equal to 5. didn't use this info $-1\frac{1}{2}$

Calculate the determinants of the following matrices.

(a) $B = \begin{bmatrix} a & b & c \\ 4d & 4e & 4f \\ -g & -h & -i \end{bmatrix}$ $4R_2$ $-R_3$

so $\det(B) = 4(-1)\det(A)$

$= -20$

1

(b) $C = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$ $R_1 \leftrightarrow R_3$

so $\det(C) = -\det(A)$

$= -5$

1

(c) $D = \begin{bmatrix} a+5d & b+5e & c+5f \\ d & e & f \\ g & h & i \end{bmatrix}$ $R_1 + 5R_2$

so $\det(C) = \det A$

$= 5$

1

2. (6 points) Find all of the eigenvalues of the matrix A over the complex numbers. Give bases for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 + 4 = 0$$

$$(1-\lambda)^2 = -4$$

$$1-\lambda = \pm\sqrt{-4} = \pm 2i$$

$$\lambda = 1 \pm 2i$$

2

or

$$1 - 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

eigenvectors

for $\lambda_1 = 1 + 2i$

$$(A - \lambda I)\vec{x} = 0$$

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow x - iy = 0$$

1 1

free variable so let $y = t$

$$\begin{cases} x = it \\ y = t \end{cases} \quad \text{so} \quad \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

1

and since if A is real, eigenvalues and eigenvectors come in complex conjugate pairs,

$$\boxed{\begin{array}{lll} \lambda_1 = 1 + 2i & \text{with} & \vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \\ \lambda_2 = 1 - 2i & \text{with} & \vec{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \end{array}}$$

1

3. (6 points) Consider the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) State the algebraic and geometric multiplicity of each eigenvalue by filling in the following table. Show any necessary work below.

eigenvalue	algebraic multiplicity	geometric multiplicity
2	1	1
1	2	2

1

(b) Is the matrix diagonalizable? Explain briefly.

yes, because alg mult = geo mult for each eigenvalue 1

find eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda = \underbrace{2}_{\text{alg mult } 1}, \underbrace{1, 1}_{\text{mult } 2}$$

1

$\lambda_1 = 2$ will have a single eigenvector and geo mult = 1

$\lambda_2 = 1$: $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1

1

y, z free variables

so two eigenvectors

geo mult = 2

1

4. (4 points) For each of the following matrices, state whether it is orthogonal. If it is, find the associated inverse. If it isn't, give a reason for why it's not.

(a) $\begin{array}{cc} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \\ v_1 \quad v_2 \end{array}$

$\vec{v}_1 \cdot \vec{v}_2 = \left(\frac{1}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)$

$\neq 0$

so $\boxed{\text{no}}$

1

or, show $AA^T \neq I$

or, show $A^T \neq A^{-1}$

(b) $\begin{array}{c} \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} \\ v_1 \quad v_2 \end{array}$

$\vec{v}_1 \cdot \vec{v}_2 = 0$ but

$\|\vec{v}_1\| = \sqrt{(-4)^2 + (-3)^2} \neq 1$

so $\boxed{\text{no}}$

1

(c) $\begin{array}{c} \begin{bmatrix} \frac{5}{13} & 0 & -\frac{12}{13} \\ \frac{12}{13} & 0 & \frac{5}{13} \\ 0 & 1 & 0 \end{bmatrix} \\ v_1 \quad v_2 \quad v_3 \end{array}$

$\vec{v}_1 \cdot \vec{v}_2 = 0$ and $\|v_1\| = \|v_2\| = \|v_3\| = 1$

$\vec{v}_1 \cdot \vec{v}_3 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = 0$

1

so matrix is orthogonal, with $A^T = A^{-1}$

so inverse is $\begin{bmatrix} 5/13 & 12/13 & 0 \\ 0 & 0 & 1 \\ -12/13 & 5/13 & 0 \end{bmatrix}$

or show $AA^T = A^TA = I$

or show $A^T = A^{-1}$

5. (6 points) Consider the subspace spanned by the following set of vectors. Use the Gram-Schmidt process to obtain an orthogonal basis for this subspace.

$$\left\{ \begin{array}{c} \vec{x}_1 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} \vec{x}_2 \\ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} \vec{x}_3 \\ \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \end{array} \right\}$$

Note: You do not need to normalize your answers.

let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ 1

then $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{-1+0-1+0}{1+1+1+0} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \\ 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$ 2

and $\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{-1+0+0+0}{1+1+1+0} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{1+0+0-3}{1+4+1+9} \right) \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{15} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 3/5 \\ 1/5 \\ -3/5 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -4 \\ 3 \\ 1 \\ -3 \end{bmatrix}$$ 3

basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \\ -3 \end{bmatrix} \right\}$