Date: Fall 2021

Name: Solution Set

Instructor: Patricia Wrean

Math 251
Test 1

Total = $\overline{}$ 20

Show your work. All of the work on this test must be your own.

GOOD LUCK!

1. (5 points) Consider the triangle ABC where

$$A = (2,2,1), \quad B = (-1,3,-1), \quad C = (0,4,1).$$

(a) Calculate the area of this triangle.

(b) Is this triangle a right triangle? Explain your reasoning.

forgot factor of ½ $\left(-\tfrac{1}{2}\right)$

each minor calculation error $\left(-\tfrac{1}{2}\right)$

a) $\vec{AB} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$ $\quad \vec{AC} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ $\quad \vec{BC} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\vec{AB} \times \vec{AC} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -3 & 1 & -2 & -3 & 1 \\ -2 & 2 & 0 & -2 & 2 \end{array} \right.$$

3

$$= 4\hat{j} - 6\hat{k} + 2\hat{k} + 4\hat{i}$$

$$= 4\hat{i} + 4\hat{j} - 4\hat{k}$$

$$\text{area} = \tfrac{1}{2} \| \vec{AB} \times \vec{AC} \| = \tfrac{1}{2}\sqrt{4^2 + 4^2 + (-4^2)}$$

$$\boxed{\begin{aligned} &= \tfrac{1}{2}\sqrt{48} \\ &= 2\sqrt{3} \\ &\approx 3.464 \end{aligned}}$$ any of these

b) $\vec{AB} \cdot \vec{AC} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = 6 + 2 + 0 \neq 0$

only checked one angle (-1)

2

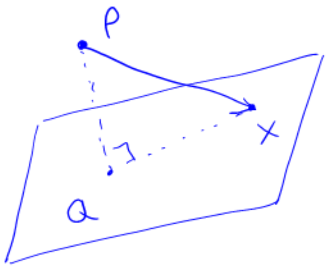
$\vec{AB} \cdot \vec{BC} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = -3 + 1 - 4 \neq 0$

$\vec{AC} \cdot \vec{BC} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = -2 + 2 + 0 = 0$

so $\vec{AC} \perp \vec{BC}$

and $\boxed{\text{yes, } \triangle ABC \text{ is a right triangle}}$

2. (5 points) Consider the point $P = (4, -2, 1)$ and the plane $3x + 2y - z = 5$. Find the point Q in the plane that is closest to P.



let X be another point on the plane, say $(0, 0, -5)$ ½

then $\vec{PX} = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$ ½

and $\vec{N} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is normal to the plane and parallel to $\vec{PQ}$ ½

$\vec{PQ} = \text{proj}_{\vec{N}}(\vec{PX})$ ½

$$= \frac{\vec{N} \cdot \vec{PX}}{\vec{N} \cdot \vec{N}} \vec{N}$$

$$= \frac{-12+4+6}{9+4+1} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = -\frac{2}{14} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$ 1

$\vec{PQ} = \vec{Q} - \vec{P}$

so $\vec{Q} = \vec{P} + \vec{PQ} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/7 \\ -2/7 \\ 1/7 \end{bmatrix}$ 1

$$= \begin{bmatrix} 25/7 \\ -16/7 \\ 8/7 \end{bmatrix}$$

optional check:
note: $3x + 2y - z = 5$
$3\left(\frac{25}{7}\right) + 2\left(\frac{-16}{7}\right) - \frac{8}{7} = 5$
$\frac{75 - 32 - 8}{7} = 5$
$\frac{35}{7} = 5$ ✓

and $\boxed{Q = \left(25/7, -16/7, 8/7\right)}$ 1

$\approx (3.57, -2.29, 1.14)$

3. (5 points) Consider the line that goes through the point P and has direction vector $\mathbf{v}$ where

$$P = (1, -1, 1) \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

(a) At what point does this line intersect the xy-plane?

(b) What angle does this line make with the xy-plane?

a) parametric equations for the line: $\begin{cases} x = t + 1 \\ y = -3t - 1 \\ z = 2t + 1 \end{cases}$ 1

if this but nothing else, +½

2

the xy-plane has $z = 0$

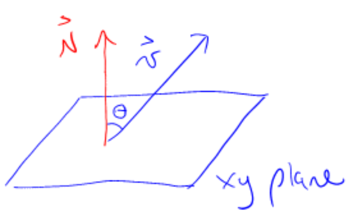
so $2t + 1 = 0$ and $t = -\frac{1}{2}$ ½

then $x = -\frac{1}{2} + 1 = \frac{1}{2}$

$y = -3(-\frac{1}{2}) - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

$\boxed{\text{point of intersection is } (\tfrac{1}{2}, \tfrac{1}{2}, 0)}$ ½

b)



if θ is the angle that vector $\vec{v}$ makes with the normal to the plane, then $(90° - \theta)$ is the angle that vector $\vec{v}$ makes with the plane 1

$$\cos\theta = \frac{\vec{N} \cdot \vec{v}}{\|\vec{N}\| \, \|\vec{v}\|}$$

where $\vec{N} = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

3

$$= \frac{2}{\sqrt{14}}$$ 1

$\vec{N} \cdot \vec{v} = 0 + 0 + 2 = 2$

$\|\vec{N}\| = 1$

$\|\vec{v}\| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$

$\theta = 57.7°$ or 1.007 rads

and angle we want is $90° - \theta$ or

$\boxed{32.3° \text{ or } 2.135 \text{ rads}}$ 1

note: other methods possible (see me for details)

4. (5 points) Consider the following systems.

(a) Use Gauss-Jordan elimination to find all solutions of the following linear system. Write your answer in parametric form. Clearly show your steps, including your row operations.

$$\begin{cases} w & -2x & +3y & & = & 4 \\ 3w & -6x & +2y & +7z & = & -2 \end{cases}$$

not full RREF

$-\frac{1}{2}$

3

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & 4 \\ 3 & -6 & 2 & 7 & -2 \end{array}\right] \xrightarrow[R_2-3R_1]{} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & 4 \\ 0 & 0 & -7 & 7 & -14 \end{array}\right]$$

$$-\tfrac{1}{7}R_2 \quad \left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

$$R_1 - 3R_3 \quad \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

$$\begin{array}{cccc} w & x & y & z \end{array}$$

2

free variables

let $x = s$

$z = t$

then $w - 2s + 3t = -2$

$y - t = 2$

$$\boxed{\begin{cases} w = 2s - 3t - 2 \\ x = s \\ y = t + 2 \\ z = t \end{cases}}$$

1

(b) For what values of h and k does the following system have one unique solution?

$$\begin{cases} x & +3y & = & 5 \\ 2x & +hy & = & k \end{cases}$$

2

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & h & k \end{array}\right] \xrightarrow[R_2-2R_1]{} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & h-6 & k-10 \end{array}\right]$$

1

for there to be one unique solution, this entry cannot be zero

note: $k-10$ can be any value

(no restrictions on k)

$\frac{1}{2}$ $\frac{1}{2}$

$$\boxed{h \neq 6, \quad k \in \mathbb{R}}$$