Date: Fall 2021

Name: Solution Set

Instructor: Patricia Wrean

Math 251
Test 2

Total = $\frac{\quad}{20}$

Show your work. All of the work on this test must be your own.

GOOD LUCK!

1. (4 points) Consider the matrices $A = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix}$.

Evaluate the following, if they exist. If they do not exist, say so.

(a) AB

$$AB = \underset{2\times 1}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} 3 & 5 \end{bmatrix}} = \begin{bmatrix} 6 & 10 \\ 12 & 20 \end{bmatrix}$$

(b) BA

$$BA = \underset{1\times 2}{\begin{bmatrix} 3 & 5 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} = \begin{bmatrix} 26 \end{bmatrix}$$

(c) B^{-1} does not exist (a matrix must be square to be invertible)

(d) C^2

$$C^2 = \begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -4 & 13 \end{bmatrix}$$

2. (2 points) Set up BUT DO NOT SOLVE a system of equations that will allow you to balance the following chemical equation.

$$Na_2CO_3 + C + N_2 \to NaCN + CO$$

$$x_1 Na_2CO_3 + x_2 C + x_3 N_2 \to x_4 NaCN + x_5 CO$$

Na: $2x_1 = x_4$

C: $x_1 + x_2 = x_4 + x_5$

O: $3x_1 = x_5$

N: $2x_3 = x_4$

$$\boxed{\text{so system is } \left\{ \begin{array}{rcl} 2x_1 \qquad\qquad - x_4 & = & 0 \\ x_1 + x_2 \qquad - x_4 - x_5 & = & 0 \\ 3x_1 \qquad\qquad\qquad - x_5 & = & 0 \\ 2x_3 - x_4 \qquad & = & 0 \end{array} \right.}$$

3. (5 points) Solve the following system using the LU method.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ 9 \\ 7 \end{bmatrix}}_{\vec{b}}$$

so $L\underbrace{U\vec{x}}=\vec{b}$

let $\vec{y}=U\vec{x}$

then $L\vec{y}=\vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 7 \end{bmatrix}$$

so $y_1 = 9$

$2y_1 + y_2 = 9$, $18 + y_2 = 9$, $y_2 = -9$

$-3y_1 - 2y_2 + y_3 = 7$, $-27 + 18 + y_3 = 7$, $y = 16$

and $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 16 \end{bmatrix}$$

where $z = 16$

$3y = -9$, $y = -3$

$2x - y + 4z = 9$, $2x + 3 + 64 = 9$,

$2x = -58$, $x = -29$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -29 \\ -3 \\ 16 \end{bmatrix}}$$

4. (4 points) Consider the following 4×4 matrix A.

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 3 & 6 & 4 & 3 \\ 2 & 4 & 2 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$A^T = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 4 \\ 1 & 2 & 4 & 2 \\ 1 & 1 & 3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give the values of Rank(A) and Nullity(A).

Rank$(A) = 3$ (= number of leading ones in RREF)

Nullity$(A) = 1$ (= number of free variables in RREF)

(b) Find a basis for Row(A) in terms of the rows of A.

basis for $\text{Col}(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 3 \end{bmatrix} \right\}$

so basis for $\text{Row}(A) = \{ [1 \ 2 \ 1 \ 1], [1 \ 2 \ 2 \ 1], [2 \ 4 \ 2 \ 3] \}$

(c) Find a basis for Col(A) in terms of the columns of A.

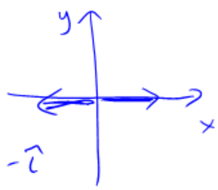
basis for $\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}$

5. (5 points) Consider the transformation T from R^2 to R^2. T first reflects a vector about the y-axis, then rotates it by 30° clockwise, then reflects it in the line $y = x$.

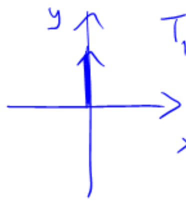
Find the standard matrix A associated with this transformation and simplify your answer.

Hint: $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

reflect about y-axis:



$T_1(\hat{i}) = -\hat{i}$



$T_1(\hat{j}) = \hat{j}$

so $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

rotate by 30° clockwise:

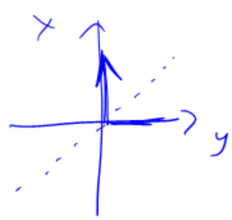
$\theta = -30°$

$A_2 = R_{(-30°)} = \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix}$

$= \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

reflect in $y = x$:

$T_3(\hat{i}) = \hat{j}$



$T_3(\hat{j}) = \hat{i}$

$A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

finally $A = A_3 A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$