Date: Fall 2021

Instructor: Patricia Wrean

Name: Solution Set

Math 251
Test 3

Total = $\overline{}$ 20

Show your work. All of the work on this test must be your own.

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u}$$

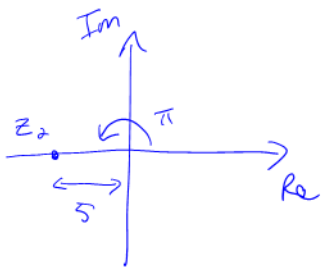
GOOD LUCK!

1. (5 points) Consider the following complex numbers.

$$z_1 = 3 - i,\ z_2 = 5e^{i\pi},\ z_3 = 4e^{i2\pi/3}$$

Evaluate the following. You may leave your answer in either rectangular or polar form, your choice. If using polar form, your angles should satisfy $0 \le \theta < 2\pi$ or $0 \le \theta < 360°$, as appropriate.

(a) $z_1 + z_2$



$z_2 = -5$ 1

so $z_1 + z_2 = (3 - i) + (-5)$

$\boxed{= -2 - i}$ 1

(b) $z_2 z_3$

$$z_2 z_3 = (5e^{i\pi})(4e^{i2\pi/3})$$

$$= 20\, e^{i(\pi + 2\pi/3)}$$

$$\boxed{= 20\, e^{i5\pi/3}}$$ 1

(c) z_3^5

$$z_3^5 = (4e^{i2\pi/3})$$

$$= 4^4\, e^{i10\pi/3}$$ ← need reference angle in $[0, 2\pi)$

$\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}$

$$\boxed{= 1024\, e^{i4\pi/3}}$$

1 1

2. (3 points) Use Cramer's Rule to solve the following system of linear equations.

$$\begin{cases} 2x + 7y = -5 \\ -x + hy = 1 \end{cases}$$

Note: your answer will be in terms of h.

1

$A = \begin{bmatrix} 2 & 7 \\ -1 & h \end{bmatrix}$ $\det(A) = 2h + 7$

$A_1(\vec{b}) = \begin{bmatrix} -5 & 7 \\ 1 & h \end{bmatrix}$ $\det(A_1(\vec{b})) = -5h - 7$

$A_2(\vec{b}) = \begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix}$ $\det(A_2(\vec{b})) = 2 - 5 = -3$

1

then $x = \dfrac{\det(A_1(\vec{b}))}{\det(A)} = \dfrac{-5h-7}{2h+7}$

$y = \dfrac{\det(A_2(\vec{b}))}{\det(A)} = \dfrac{-3}{2h+7}$

$$\boxed{(x, y) = \left(\frac{-5h-7}{2h+7}, \frac{-3}{2h+7} \right)}$$

1

3. (6 points) Consider the matrix A given below. Find diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. Note: you do not need to calculate P^{-1}.

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

first find eigenvalues

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$(2-\lambda)^2(3-\lambda) = 0$

$\lambda = 2, 2, 3$ 1

then find eigenvectors: solve $(A - \lambda I)\vec{x} = 0$

for $\lambda_1 = 2$, $A - \lambda I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 2

let $x = s$ let $z = t$

$\begin{cases} x = s \\ y = -t \\ z = t \end{cases}$ so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda_2 = 3$, $A - \lambda I = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 1

let $z = t$

$\begin{cases} x = 2t \\ y = 0 \\ z = t \end{cases}$ so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\text{finally, } P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ with } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

1 1

4. (6 points) Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{u}$ below and let subspace $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u} \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set.

(b) Find the associated orthonormal set for $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Find $\text{proj}_W(\mathbf{u})$ and $\text{perp}_W(\mathbf{u})$.

a) $\vec{v}_1 \cdot \vec{v}_2 = 1 + 0 - 1 + 0 = 0$ so $\vec{v}_1 \perp \vec{v}_2$ 1

b) orthonormal set $= \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right.$ 1

c) $\text{proj}_W(\vec{u}) = \text{proj}_{\vec{v}_1}(\vec{u}) + \text{proj}_{\vec{v}_2}(\vec{u})$

$$= \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{3-2}{6} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$ 3

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 1/6 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7/6 \\ -1 \\ 5/6 \\ 1/3 \end{bmatrix}$$

$\text{perp}_W(\vec{u}) = \vec{u} - \text{proj}_W(\vec{u}) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 7/6 \\ -1 \\ 5/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 11/6 \\ 1 \\ -5/6 \\ -4/3 \end{bmatrix}$ 1