Date: Fall 2022

Name: Solution Set

Instructor: Patricia Wrean

Math 251
Test 2

Total = $\overline{}\,20$

Show your work. All of the work on this test must be your own.

GOOD LUCK!

1. (4 points) Consider the matrix $A = \begin{bmatrix} 2 & 3 \end{bmatrix}$.

   Evaluate the following, if they exist. If they do not exist, say so.

   (a) AA^T $= \underbrace{\begin{bmatrix} 2 & 3 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 2^2+3^2 \end{bmatrix}}_{1\times 1} = \begin{bmatrix} 13 \end{bmatrix}$

   (b) $A + A^T$ $= \begin{bmatrix} 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ = undefined or DNE

   (c) A^TA $= \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} \underbrace{\begin{bmatrix} 2 & 3 \end{bmatrix}}_{1\times 2} = \underbrace{\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}}_{2\times 2}$

   (d) A^2 $= \underbrace{\begin{bmatrix} 2 & 3 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 2 & 3 \end{bmatrix}}_{1\times 2}$ = undefined or DNE

2. (2 points) Solve the given matrix equation for X. Assume that all matrices are invertible.

$$(BX)^{-1} = B^{-1}A$$

method 1:

$BX = (B^{-1}A)^{-1}$

$= A^{-1}B$

$B^{-1}BX = B^{-1}A^{-1}B$

$\boxed{X = B^{-1}A^{-1}B}$

method 2:

$X^{-1}B^{-1} = B^{-1}A$

$X^{-1}B^{-1}B = B^{-1}AB$

$X^{-1} = B^{-1}AB$

$X = (B^{-1}AB)^{-1}$

$\boxed{X = B^{-1}A^{-1}B}$

3. (4 points) Find an LU factorization of the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{R_3 + 2R_2}$$

$$U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

2

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

$R_2 + 2R_1$

$R_3 - R_1$, $R_3 - 2R_2$

2

$$\text{so } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

4. (5 points) Consider the following matrix A.

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 2 & 1 & 6 & 1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

free variables

(a) Give the values of Rank(A) and Nullity(A).

1 Rank(A) = 3 (# of leading ones in RREF)

1 Nullity(A) = 2 (# of free variables in RREF)

(b) Find a basis for Null(A).

RREF gives
$$\begin{cases} x_1 + x_3 - x_5 = 0 \\ x_2 + 2x_3 + 3x_5 = 0 \\ x_4 + 4x_5 = 0 \end{cases}$$
1

now let $x_3 = s$
$x_5 = t$

$$\begin{cases} x_1 = -s + t \\ x_2 = -2s - 3t \\ x_3 = s \\ x_4 = -4t \\ x_5 = t \end{cases}$$
1

or $\vec{x} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

a basis for Null(A) is $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ 1

5. (5 points) Find the equation of the parabola that passes through the points $(1, -2)$, $(-2, 10)$, and the origin.

   Set up and solve a system to find this parabola, writing your answer in the form $y = ax^2 + bx + c$. Use Gauss-Jordan elimination and be sure to specify which row operations you are using.

$y = ax^2 + bx + c$

for $(1,-2)$: $-2 = a(1)^2 + b(1) + c$

$(-2,10)$: $10 = a(-2)^2 + b(-2) + c$

$(0,0)$: $0 = a(0)^2 + b(0) + c$

so $\begin{cases} -2 = a + b + c \\ 10 = 4a - 2b + c \\ 0 = c \end{cases}$ 2

system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 10 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\downarrow$

$R_1 - R_3$, $R_2 - R_3$ $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 4 & -2 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\downarrow$

$R_2 - 4R_1$ $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & -6 & 0 & 18 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\downarrow$

$-\frac{1}{6} R_2$ $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\downarrow$

$R_1 - R_2$ $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$a = 1$

$b = -5$

$c = 1$

2

no Gauss Jordan -2

did GJ but not to RREF -1

So $\boxed{y = x^2 - 3x}$ 1

optional check:

$(0,0)$: $0 = 0^2 - 0$ ✓

$(1,-2)$: $-2 = 1^2 - 3$ ✓

$(-2,10)$: $10 = 4 - 3(-2)$ ✓

-2

if didn't read the question and did only

$$-2 = a + b + c$$
$$10 = 4a - b + c$$

get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 10 \end{array}\right]$$
$$\downarrow R_2 - 4R_1$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 18 \end{array}\right]$$
$$\downarrow -\tfrac{1}{6} R_2$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & \tfrac{1}{2} & -3 \end{array}\right]$$
$$\downarrow R_1 - R_2$$
$$\left[\begin{array}{ccc|c} 1 & 0 & \tfrac{1}{2} & 1 \\ 0 & 1 & \tfrac{1}{2} & -3 \end{array}\right]$$

then $a + \tfrac{1}{2}c = 1$

$b + \tfrac{1}{2}c = -3$

so let $c = t$

$$\begin{cases} a = 1 - \tfrac{1}{2}t \\ b = -3 - \tfrac{1}{2}t \\ c = t \end{cases}$$

if put in an arbitrary value for t -1

and if you insist

$$\boxed{y = \left(1 - \tfrac{1}{2}t\right)x^2 + \left(-3 - \tfrac{1}{2}t\right)x + t}$$