Date: Fall 2022

Name: Solution Set

Instructor: Patricia Wrean

Math 251
Test 3

Total = $\overline{}$ 20

Show your work. All of the work on this test must be your own.

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

GOOD LUCK!

1. (5 points) Consider the transformation T which rotates a vector in $\mathbb{R}^2$ clockwise by $135°$ and then reflects it about the line $y = -x$.

$\theta = -135°$

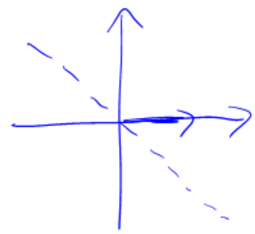
$(-\frac{1}{2})$ if $+135°$

(a) Find the standard matrix A for this transformation.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{135°} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad 1$$

$$A_{reflect} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad 1$$



$T_{reflect}(\hat{i}) = -\hat{j}$

$T_{ref}(\hat{j}) = -\hat{i}$

then $A = A_{reflect} R_{135°}$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad 1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad 1$$

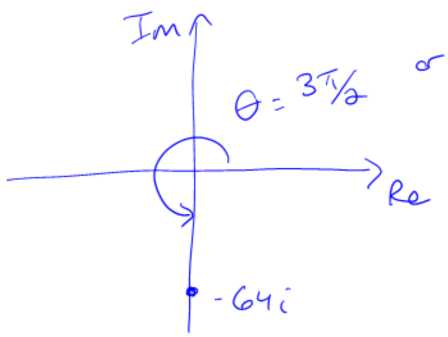
(b) Now evaluate $T(\mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

$$T(\vec{v}) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix} \quad 1$$

2. (4 points) Find all values of z such that $z^6 = -64i$.

Give exact answers. You may leave your answer in either rectangular or polar form, your choice. If using polar form, your angles should satisfy $0 \le \theta < 2\pi$ or $0 \le \theta < 360°$, as appropriate.



or $270°$ (or $-\pi/2$)

$$z^6 = -64i$$

$$= 64 e^{i\,3\pi/2} \qquad 1$$

$$z = \left(64 e^{i\,3\pi/2}\right)^{1/6}$$

$$= 2e^{i\,3\pi/12} = 2e^{i\pi/4} \qquad 1$$

so first root is at $2e^{i\pi/4}$
– there are 6 in total, $\frac{2\pi}{6} = \pi/3$ rads apart 1

then $z = 2e^{i\pi/4},\ 2e^{i(\pi/4 + \pi/3)},\ 2e^{i(\pi/4 + 2\pi/3)},\ 2e^{i(\pi/4 + \pi)},$
$2e^{i(\pi/4 + 4\pi/3)},\ 2e^{i(\pi/4 + 5\pi/3)}$

1 $\quad = 2e^{i\pi/4},\ 2e^{i7\pi/12},\ 2e^{i11\pi/12},\ 2e^{i5\pi/4},\ 2e^{i19\pi/12},\ 2e^{i\,23\pi/12}$

or (easier)

$z = 2\angle 45°,\ 2\angle 105°,\ 2\angle 165°,\ 2\angle 225°,\ 2\angle 285°,\ 2\angle 345°$

(you could do rectangular if you really insisted, but getting the exact form is long and annoying)

3. (2 points) Compute the determinant of the following matrix.

if omitted $-\frac{1}{2}$

$$A = \begin{bmatrix} 2 & 0 & 3 & 5 \\ -1 & -2 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 3 & -3 & k & 0 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Your answer should be in terms of k.

method of minors - expand about row 3:

$$\det(A) = -2 \begin{vmatrix} 2 & 3 & 5 \\ -1 & 0 & 3 \\ 3 & k & 0 \end{vmatrix} \begin{matrix} 2 & 3 \\ -1 & 0 \\ 3 & k \end{matrix}$$

$$= -2(0 + 27 - 5k - 0 - 6k - 0)$$

$$= -2(27 - 11k)$$

$$= -54 + 22k$$

4. (5 points) Consider the matrix A given below.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

(a) Find diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$.

find eigenvalues by $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \qquad = (1-\lambda)(3-\lambda) - 8$$
$$= 3 - 4\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 4\lambda - 5$$
$$= (\lambda - 5)(\lambda + 1)$$

so $\boxed{\lambda = 5, -1}$ 1

then $\lambda = 5$, solve $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{cases} x = \frac{1}{2}t \\ y = t \end{cases}$ eigenvector is $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 1

$\lambda = -1$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{cases} x = -t \\ y = t \end{cases}$ eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ 1

$\boxed{\text{so } D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}$ 1

(b) Using your result for part (a), calculate A^6.

$$A^6 = PD^6P^{-1}$$
$$= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5^6 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 5^6 & -1 \\ 2 \cdot 5^6 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 5^6 + 2 & 5^6 - 1 \\ 2 \cdot 5^6 - 2 & 2 \cdot 5^6 + 1 \end{bmatrix} = \begin{bmatrix} 5209 & 5208 \\ 10416 & 10417 \end{bmatrix}$$

1

5. (4 points) Consider the set of vectors $\mathcal{B}$ below and also the vector $\mathbf{u}$. $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}} \right\} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$$

(a) Show that $\mathcal{B}$ is an **orthogonal** basis of $\mathbb{R}^3$.

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$$

2

(b) Using the fact that $\mathcal{B}$ is an **orthogonal** basis, express the vector $\mathbf{u}$ as a linear combination of the vectors in $\mathcal{B}$.

then $\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$

where $c_1 = \dfrac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \dfrac{-1+5}{2} = 2$

$$c_2 = \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{-1-5-3}{1+1+1} = -3$$

$$c_3 = \frac{\vec{u} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{-1-5+6}{1+1+4} = 0$$

2

$\vec{u} = 2\vec{v}_1 - 3\vec{v}_2$ or $[u]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$