

31.7 and 31.8 Higher-Order Homogeneous DEs

DE	$ay'' + by' + cy = 0$	$ay''' + by'' + cy' + dy = 0$
Auxilliary Equation	$am^2 + bm + c = 0$	$am^3 + bm^2 + cm + d = 0$
Case 1	2 real solutions m_1 and m_2 $y = C_1e^{m_1x} + C_2e^{m_2x}$	3 real solutions m_1, m_2 and m_3 $y = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x}$
Case 2	1 real (repeated) solution m $y = (C_1 + C_2x)e^{mx}$	1 real (repeated) solution m $y = (C_1 + C_2x + C_3x^2)e^{mx}$
Case 3	2 complex solutions $\alpha \pm \beta i$ $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$	1 real solution m and 2 complex solutions $\alpha \pm \beta i$ $y = C_1e^{mx} + e^{\alpha x}(C_2 \cos \beta x + C_3 \sin \beta x)$
Case 4		2 real solutions m_1 (repeated) and m_2 $y = (C_1 + C_2x)e^{m_1x} + C_3e^{m_2x}$

In general:

1. the order of the DE = the total number of constants in the solution
 - 2nd order $\Rightarrow C_1$ and C_2
 - 3rd order $\Rightarrow C_1, C_2$ and C_3
 - 4th order $\Rightarrow C_1, C_2, C_3$ and C_4

2. the multiplicity of a linear factor = the number of constants in the corresponding part of the solution
 - $(m - 7) \Rightarrow C_1e^{7x}$
 - $(m - 7)^2 \Rightarrow (C_1 + C_2x)e^{7x}$
 - $(m - 7)^3 \Rightarrow (C_1 + C_2x + C_3x^2)e^{7x}$
 - $(m^2 + 1) = (m - i)(m + i) \Rightarrow C_1 \sin x + C_2 \cos x$
 - $(m^2 + 1)^2 = (m - i)^2(m + i)^2 \Rightarrow (C_1 + C_2x) \sin x + (C_3 + C_4x) \cos x$

	Example 1	Example 2	Example 3
DE	$y'' + y' - 2y = 0$	$y'' - 6y' + 9y = 0$	$y'' + y' + y = 0$
Auxilliary Equation	$m^2 + m - 2 = 0$ $(m + 2)(m - 1) = 0$ $m = -2, 1$	$m^2 - 6m + 9 = 0$ $(m - 3)^2 = 0$ $m = 3$ (multiplicity 2)	$m^2 + m + 1 = 0$ $m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
Solution	$y = C_1 e^{-2x} + C_2 e^x$	$y = (C_1 + C_2 x) e^{3x}$	$y = e^{-\frac{1}{2}x} \left(C_1 \sin \frac{\sqrt{3}}{2}x + C_2 \cos \frac{\sqrt{3}}{2}x \right)$

	Example 4	Example 5	Example 6
DE	$y''' - y' = 0$	$y''' + 4y'' + 4y' = 0$	$y''' + 4y' = 0$
Auxilliary Equation	$m^3 - m = 0$ $m(m + 1)(m - 1) = 0$ $m = 0, -1, 1$	$m^3 + 4m^2 + 4m = 0$ $m(m + 2)^2 = 0$ $m = 0, -2$ (mult. 2)	$m^3 + 4m = 0$ $m(m^2 + 4) = 0$ $m = 0, \pm 2i$
Solution	$y = C_1 + C_2 e^{-x} + C_3 e^x$	$y = C_1 + (C_2 + C_3 x) e^{-2x}$	$y = C_1 + C_2 \sin 2x + C_3 \cos 2x$

	Example 7	Example 8
DE	$y^{(4)} - y = 0$	$y^{(4)} - 2y''' + y'' = 0$
Auxilliary Equation	$m^4 - 1 = 0$ $(m^2 + 1)(m + 1)(m - 1) = 0$ $m = \pm i, -1, 1$	$m^4 - 2m^3 + m^2 = 0$ $m^2(m - 1)^2 = 0$ $m = 0$ (mult.2), 1 (mult. 2)
Solution	$y = C_1 \sin x + C_2 \cos x + C_3 e^{-x} + C_4 e^x$	$y = C_1 + C_2 x + (C_3 + C_4 x) e^x$