Math 252: Definitions

Differential Equation:

A **differential equation** (DE) is an equation involving derivatives (or differentials) of one or more dependent variables with respect to one or more independent variables.

In this course, we are only looking at ordinary differential equations, those with only one independent variable.

Examples of ordinary differential equations:

- $\dfrac{dy}{dx} = \sin x$
- $x^2y'' - 2xy' + 6y = 0$
- $ma = mg - kv$ where a and v are acceleration and velocity, respectively

Notation:

- $\dfrac{dy}{dt}, \dfrac{d^2y}{dt^2}, \dfrac{d^3y}{dt^3}, \ldots$ (Leibniz notation)
- $y', y'', y''', y^{(4)}, \ldots$ (prime notation, derivative with respect to x, so $y' = \dfrac{dy}{dx}$)
- $\dot{x}, \ddot{x}, \dddot{x}, \ldots$ (dot notation, derivative with respect to t, so $\dot{y} = \dfrac{dy}{dt}$ and $\ddot{y} = \dfrac{d^2y}{dt^2}$)

Solution to a DE:

The **solution** to a DE is a function that when substituted into the DE, reduces the DE to an identity. The solution itself will not contain derivatives or differentials.

Examples: Show that the following functions are solutions to the given DE.

1. $y = c\ln x$, where c is a constant; $y'\ln x - y/x = 0$
2. $y^3 - x^2 = 1$; $\dfrac{dy}{dx} = \dfrac{2x}{3y^2}$

Particular Solution:

A **particular solution** contains no parameters (unknown constants).

Example: Consider the DE $y'' + y = 0$.

- $y = C_1\sin x + C_2\cos x$ is not a particular solution, while $y = 5\sin x + 3\cos x$ is.

Implicit vs. Explicit Solution:

An **explicit solution** is one in which the dependent variable is expressed in terms of the dependent variable and constants (solved for the dependent variable). Otherwise, the solution is said to be **implicit**.

Example: Consider the DE $\frac{dy}{dx} = \frac{2x}{3y^2}$.

- $y^3 - x^2 = 1$ is an implicit solution, while $y = \sqrt[3]{x^2 + 1}$ is an explicit solution

Order of a DE:

The order of a differential equation is the order of the highest derivative in the equation. For example, if a DE contains both a first derivative and a second derivative, then the equation itself is also second-order.

Example: State the order of the following DEs.

- $y'' + 5(y')^3 + y = 0$
- $x^6 y''' - y' = e^x$
- $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = \cos \omega t$

Linear DE:

Definition

An nth order ordinary differential equation is linear if it can be written as:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + ...a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

In other words, all of the "coefficients" on y and its derivatives are only functions of x.

Example: State whether the following DEs are linear or non-linear.

1. $y'' + 8y = e^x$
2. $y' - y \sin x = 0$
3. $y^2 = y'$
4. $e^x y'' + y = 0$
5. $e^y y'' + y = 0$