Section 2.5: Full example

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solve explicitly:

$$(x^2+y^2)\,dx - xy\,dy = 0$$

homogeneous of degree 2

method 1:

$$\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$$

DE becomes

$$(x^2+u^2x^2)dx - x(ux)(u\,dx + x\,du) = 0$$

$$x^2dx + \cancel{u^2x^2dx} - \cancel{u^2x^2dx} - ux^3du = 0$$

$$x^2dx = ux^3du$$

separable

$$\int \frac{dx}{x} = \int u\,du$$

$$\ln|x| = \frac{u^2}{2} + C$$

$$\ln|x| = \frac{y^2}{2x^2} + C$$

$$\ln|x| - C = \frac{y^2}{2x^2}$$

method 2

$$\begin{cases} x = vy \\ dx = v\,dy + y\,dv \end{cases}$$

$$(v^2y^2+y^2)(v\,dy + y\,dv) - vy\cdot y\,dy = 0$$

$$v^3y^2dy + v^2y^3dv + \cancel{vy^2dy} + y^3dv - \cancel{vy^2dy} =$$

$$v^3y^2dy + v^2y^3dv + y^3dv = 0$$

$$v^3y^2dy = -(v^2y^3+y^3)dv$$

$$v^3y^2dy = -(v^2+1)y^3dv$$

$$\int \frac{dy}{y} = \int -\frac{v^2+1}{v^3}\,dv$$

$$\ln|y| = -\int\left(\frac{1}{v}+\frac{1}{v^3}\right)dv$$

$$\ln|y| = -\ln|v| + \frac{1}{2v^2} + C$$

$$\ln|y| = -\ln\left|\frac{x}{y}\right| + \frac{1}{2\left(\frac{x}{y}\right)^2} + C$$

$\ln|x| - C = \frac{y^2}{2x^2}$

$2\ln|x| - 2C = \frac{y^2}{x^2}$ $\quad c_1$

$2\ln|x| + c_1 = \frac{y^2}{x^2}$

$x^2(2\ln|x| + c_1) = y^2$

$y = \pm x\sqrt{2\ln|x| + c_1}$

$\ln|y| = -\ln\left|\frac{x}{y}\right| + \frac{1}{2\left(\frac{x}{y}\right)^2} + C$

$\cancel{\ln|y|} = -\ln|x| + \cancel{\ln|y|} + \frac{y^2}{2x^2} + C$

$\ln|x| = \frac{y^2}{2x^2} + C$

etc

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but also: method #3:

$$(x^2 + y^2)\,dx - xy\,dy = 0$$

solve for $\frac{dy}{dx}$: $\quad (x^2+y^2)\,dx = xy\,dy$

$$\frac{x^2+y^2}{xy} = \frac{dy}{dx}$$

so $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

sub in $\begin{cases} y = ux \\ \frac{dy}{dx} = u + \frac{du}{dx}\cdot x \end{cases}$

DE becomes

$$u + \frac{du}{dx}\cdot x = \frac{x^2 + u^2x^2}{}$$

$$u + \frac{du}{dx} \cdot x = \frac{x^2 + u^2 x^2}{x(ux)}$$

$$u + \frac{du}{dx} \cdot x = \frac{x^2 + u^2 x^2}{ux^2} = \frac{1 + u^2}{u} = \frac{1}{u} + u$$

$$\frac{du}{dx} \cdot x = \frac{1}{u} + u - u$$

$$\frac{du}{dx} \cdot x = \frac{1}{u}$$

$$u\,du = \frac{dx}{x}$$

which is the separable DE we
found using method #1