Math 252: Theory Behind Higher Order Linear DEs

Preliminary Theory:

if $g(x)=0$, homogeneous

$g(x)\neq 0$, nonhomogeneous

Consider the following linear n^{th}-order DE.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

For an initial-value problem (IVP), we have n initial conditions:

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

We always assume that $a_n(x), \ldots, a_1(x), a_0(x)$ are continuous and $a_n(x) \neq 0$ for all x in an interval I containing x_0.

Theorem: under these assumptions, the IVP has a unique solution.

if $a_n(x)=0$ for any point, may not have unique soln or even any soln

Homogeneous Linear DEs:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad \text{[H]}$$

The solutions of [H] satisfy the **Principle of Superposition**:

If $y_1, y_2, \ldots, y_k$ are solutions of [H], then

$$c_1y_1 + c_2y_2 + \ldots + c_ky_k$$

is also a solution for any constants $c_1, c_2, \ldots, c_k$.

Definition: A set of functions $\{y_1, y_2, \ldots, y_n\}$ definied over an initial interval I is **linearly dependent (LD)** if there exists constants $c_1, c_2, c_3, \ldots, c_k$ not all zero such that:

$$c_1y_1 + c_2y_2 + \ldots + c_ky_k = 0$$

Otherwise, they are **linearly independent (LI)**.

How do we test for LI? We can use a Wronskian:

Definition: Consider the functions $y_1, y_2, \ldots, y_n$. The Wronskian is:

$$W = W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1, & y_2, & \ldots, & y_n \\ y_1', & y_2', & \ldots, & y_n' \\ & \vdots & & \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \ldots, & y_n^{(n-1)} \end{vmatrix}$$

Theorem: If $y_1, y_2, \ldots, y_n$ are solutions of [H], then they will be LI on an interval I provided that $W \neq 0$ for all x in that interval.

Fundamental Set of Solutions:

Definition: a basis, or fundamental set of solutions, of all the solutions of [H] consists of n linearly independent solutions.

If $\{y_1, y_2, \ldots, y_n\}$ is a basis of all solutions of [H], then any other solution y of [H] can be expressed uniquely as $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$.