Explainer: Reduction of Order

Thursday, January 30, 2020 12:55 PM

Consider the 2^{nd} order linear homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0$$

in standard form, with solution $y_1(x)$.

assume that there is a second solution $y_2(x)$ and let

$$u(x) = \frac{y_2(x)}{y_1(x)}$$

then $y_2(x) = u(x)\, y_1(x)$

or just $y_2 = u y_1$

then $y_2' = u y_1' + u' y_1$

$$y_2'' = \underbrace{u y_1'' + u' y_1'}_{} + \underbrace{u' y_1' + u'' y_1}_{}$$

$$= u y_1'' + 2u' y_1' + u'' y_1$$

now, sub into DE:

$$y'' + Py' + Qy = 0$$

$$(u y_1'' + 2u' y_1' + u'' y_1) + P(u y_1' + u' y_1) + Q u y_1 = 0$$

$$\underbrace{u(y_1'' + Py_1' + Qy_1)}_{\text{this is DE, so } = 0} + u'(2y_1' + Py) + u''y_1 = 0$$

so $y_1 u'' + (2y_1' + Py_1)u' = 0$

but let $w = u'$
then $w' = u''$

and $y_1 w' + (2y_1' + Py_1)w = 0$

which is linear 1^{st} order <u>and</u> separable

rewriting: $$y_1 \frac{dw}{dx} + (2y_1' + Py_1)w = 0$$

$$\frac{dw}{dx} + \left(\frac{2y_1'}{y_1} + P\right)w = 0$$

$$\frac{dw}{w} = -\left(\frac{2y_1'}{y_1} + P\right)dx$$

$$\ln|w| = -2\ln|y_1| - \int P\,dx + C$$

$$\ln|w| + 2\ln|y_1| = -\int P\,dx + C$$

$$\ln|wy_1^2| = -\int P\,dx + C$$

$$|wy_1^2| = e^{-\int P\,dx + C}$$

$$w y_1^2 = \pm e^{-\int P dx} e^{C} \quad = C_1$$

$$w = \frac{C_1}{y_1^2} e^{-\int P dx}$$

but recall $\quad w = u' = \frac{du}{dx}$

so $\quad u = \int u' dx \quad$ or $\quad \int \frac{du}{dx} dx$

$$u = \int \frac{C_1}{y_1^2} e^{-\int P dx}$$

and lastly $\quad y_2 = u y_1$

$$= y_1 \int \frac{C_1 e^{-\int P dx}}{y_1^2}$$

and since we will be multiplying y_2 by an arbitrary constant for the general solution, we set $C_1 = 1$ and get

$$\boxed{y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx}$$