Section 4.3: Complex Roots

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suppose we have a 2^{nd} order linear homogeneous DE that when we solve the auxiliary equation, we get solutions of the form

$$m = a \pm bi \qquad \text{where } a \text{ and } b \text{ are real}$$

what do the solutions to the DE look like?
we'll need the fact that

$$e^{ix} = \cos x + i \sin x$$

now, because $m = a \pm bi$, so we have two distinct roots, then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$
$$= c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}$$
$$= c_1 e^{ax+bix} + c_2 e^{ax-bix}$$
$$= c_1 e^{ax} e^{bix} + c_2 e^{ax} e^{-bix}$$
$$= c_1 e^{ax} (\cos bx + i \sin bx) + c_2 e^{ax} (\cos(-bx)) + i \sin(-bx))$$

now recall that $\cos x$ is an even function, while $\sin x$ is odd, so $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$$y = c_1 e^{ax}(\cos bx + i \sin bx) + c_2 e^{ax}(\cos bx - i \sin bx)$$

$$= \underbrace{(c_1 + c_2)}_{\text{new constant } A} e^{ax} \cos bx + \underbrace{(c_1 - c_2)i}_{\text{new constant } B} e^{ax} \sin bx$$

$$= A e^{ax} \cos bx + B e^{ax} \sin bx$$

$$= e^{ax}(A \cos bx + B \sin bx)$$