Explainer: Higher Order Linear Homogeneous DEs with Constant Coefficients

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consider $y'' - 4y' - 21y = 0$

$D^2 y - 4Dy - 21y = 0$ using operator notation $\frac{d}{dx} = D$

$(D^2 - 4D - 21)y = 0$

$(D-7)(D+3)y = 0$

now let $u = (D+3)y$

then $(D-7)u = 0$

$\frac{du}{dx} - 7u = 0$ linear 1st order (also, separable)

$IF = e^{\int P(x)dx}$

$= e^{\int -7dx} = e^{-7x}$

$\frac{du}{dx}e^{-7x} - 7ue^{-7x} = 0$

$\int \frac{d}{dx}(ue^{-7x})dx = \int 0$

$ue^{-7x} = C$

$u = Ce^{7x}$

but $u = (D+3)y$

$Ce^{7x} = \frac{dy}{dx} + 3y$

$$\frac{dy}{dx} + 3y = Ce^{7x}$$

1^{st} order linear (and separable)

$$IF = e^{\int P(x)dx} = e^{\int 3dx}$$

$$= e^{3x}$$

$$\frac{dy}{dx}e^{3x} + 3ye^{3x} = Ce^{7x}e^{3x}$$

$$\int \frac{d}{dx}\left(ye^{3x}\right)dx = \int Ce^{10x}dx$$

$$ye^{3x} = \frac{C}{10}e^{10x} + C_1$$

$$y = \underbrace{\frac{C}{10}}_{C_2}e^{7x} + C_1e^{-3x}$$

$$\boxed{y = C_2e^{7x} + C_1e^{-3x}}$$

generalizing: $(D-m_1)(D-m_2)y = 0$

let $u = (D-m_2)y$

$$(D-m_1)u = 0$$

$$IF = e^{\int P(x)dx} = e^{-m_1x}$$

$$ue^{-m_1x} = C$$

$$u = Ce^{m_1x}$$

then $$u = (D - m_2)y$$

$$Ce^{m_1 x} = \frac{dy}{dx} - m_2 y$$

$$\frac{dy}{dx} - m_2 y = Ce^{m_1 x} \qquad IF = e^{-m_2 x}$$

$$ye^{-m_2 x} = \int Ce^{m_1 x} e^{-m_2 x}\,dx$$

$$y e^{-m_2 x} = \frac{C}{m_1 - m_2} e^{(m_1 - m_2)x} + C_1$$

$$y = \frac{C}{m_1 - m_2} e^{m_1 x} + C_1 e^{m_2 x}$$

$$\boxed{y = C_2 e^{m_1 x} + C_1 e^{m_2 x}}$$

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what about the case where there's only one distinct real root to aux eqn?

$$y'' - 6y' + 9y = 0$$

$$D^2 y - 6Dy + 9y = 0$$

$$(D^2 - 6D + 9)\,y = 0$$

$$(D-3)(D-3)\,y = 0$$

let $u = (D-3)y$

$$(D-3)\,u = 0$$

$$(D-3)v = 0$$

$$\frac{dv}{dx} - 3v = 0$$

$$IF = e^{\int P(x)dx}$$
$$= e^{\int -3dx}$$
$$= e^{-3x}$$

$$e^{-3x}\frac{dv}{dx} - 3ve^{-3x} = 0$$

$$\int \frac{d}{dx}(ve^{-3x})dx = \int 0$$

$$ve^{-3x} = C$$

$$v = Ce^{3x}$$

but $v = (D-3)y$

$$Ce^{3x} = \frac{dy}{dx} - 3y$$

$$\frac{dy}{dx} - 3y = Ce^{3x}$$

$$IF = e^{\int P(x)dx}$$
$$= e^{\int -3dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3ye^{-3x} = C$$

$$\int \frac{d}{dx}(ye^{-3x})dx = \int C\,dx$$

$$ye^{-3x} = Cx + C_1$$

$$\boxed{y = Cxe^{-3x} + C_1e^{-3x}}$$