Explainer: Variation of Parameters

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Consider the DE

$$y'' + P(x)\,y' + Q(x)\,y = f(x)$$

in which y_1 and y_2 are solutions to the associated homogeneous DE $y'' + P(x)\,y' + Q(x)\,y = 0$

let's suppose that we can write y_p as

$$y_p = u_1(x)\cdot y_1(x) + u_2(x)\cdot y_2(x)$$

where $u_1(x)$ and $u_2(x)$ are functions that we'd like to find.

Then

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_1'' y_1 + u_1' y_1' + u_2 y_2'' + u_2' y_2' + u_2'' y_2 + u_2' y_2'$$

$$= u_1 y_1'' + 2u_1' y_1' + u_1'' y_1 + u_2 y_2'' + 2u_2' y_2' + u_2'' y_2$$

Substitute back into DE:

$$y_p'' + P(x)\,y_p' + Q(x)\,y_p = f(x)$$

$$u_1 y_1'' + 2u_1' y_1' + u_1'' y_1 + u_2 y_2'' + 2u_2' y_2' + u_2'' y_2$$
$$+ P\left(u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2\right)$$
$$+ Q\left(u_1 y_1 + u_2 y_2\right) = f(x)$$

regrouping gives

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$$u_1\left[\cancel{y_1'' + Py_1'} + Qy_1\right] + u_2\left[y_2'' \cancel{+ Py_2'} + Qy_2\right]$$

$$+ 2u_1'y_1' + 2u_2'y_2' + Pu_1'y_1 + Pu_2'y_2$$

$$+ u_1''y_1 + u_2''y_2 \qquad = f(x)$$

but since y_1 and y_2 solve the associated homogeneous DE,

$$y_1'' + Py_1' + Qy_1 = 0$$

$$\text{and} \quad y_2'' + Py_2' + Qy_2 = 0$$

so

$$2u_1'y_1' + 2u_2'y_2' + Pu_1'y_1 + Pu_2'y_2$$

$$+ u_1''y + u_2''y_2 \qquad = f(x)$$

regrouping gives

$$\left(u_1'y_1' + u_1''y_1\right) + \left(u_2'y_2' + u_2''y_2\right)$$

$$+ P\left(u_1'y_1 + u_2'y_2\right) + u_1'y_1' + u_2y_2' = f(x)$$

and notice that $\qquad \dfrac{d}{dx}\left(u_1y_1' + u_2y_2'\right) = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$

so we get

$$\frac{d}{dx}\left(u_1y_1' + u_2y_2'\right) + P\left(u_1'y_1 + u_2'y_2\right)$$

$$+ u_1'y_1' + u_2y_2' = f(x)$$

Suppose we choose that

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$$u_1 y_1' + u_2 y_2' = 0$$

then
$$\frac{d}{dx}\left(u_1 y_1' + u_2 y_2' \right) = 0 \quad \text{also}$$

and the above eqn reduces to

$$u_1' y_1' + u_2 y_2' = f(x)$$

which gives us two equations in u_1 and u_2:

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = f(x)$$

which we solve using Cramer's rule