

Explainer: Why no +C in Variation of Parameters

Friday, February 14, 2020 9:14 AM

consider the DE

$$y'' - 2y' + y = \frac{e^x}{x}$$

$$y_c = \begin{aligned} y'' - 2y' + y &= 0 \\ m^2 - 2m + 1 &= 0 \\ (m-1)^2 &= 0 \\ m &= 1, 1 \end{aligned}$$

$$y_c = (C_1 + C_2 x) e^x$$

$$y_1 = e^x,$$

$$y_2 = x e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix} = x e^{2x} + e^{2x} - x e^{2x} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x e^x \\ \frac{e^x}{x} & x e^x + e^x \end{vmatrix} = -e^{2x}$$

$$W_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix} = \frac{1}{x} e^{2x}$$

$$U_1 = \int \frac{W_1}{W} dx = \int \frac{-e^{2x}}{e^{2x}} dx = \int -dx = -x + C$$

$$U_2 = \int \frac{W_2}{W} dx = \int \frac{\frac{1}{x} e^{2x}}{e^{2x}} dx = \int \frac{1}{x} dx = \ln|x| + 0$$

$$y_p = U_1 y_1 + U_2 y_2$$

$$= (-x) e^x + (\ln|x|) x e^x$$

$$y_p = (-x + c)e^x + (\ln|x| + D)xe^x$$

$$\begin{aligned}
 y &= y_c + y_p \\
 &= (C_1 + C_2 x)e^x + (-x + c)e^x + (\ln|x| + D)xe^x \\
 &= C_1 e^x + C_2 x e^x - x e^x + c e^x + x e^x \ln|x| + D x e^x \\
 &= \underbrace{(C_1 + c)}_{\substack{\text{new} \\ \text{constant} \\ C_3}} e^x + \underbrace{(C_2 + D)}_{\substack{\text{new} \\ \text{constant} \\ C_4}} x e^x - x e^x + x e^x \ln|x|
 \end{aligned}$$

$$= C_3 e^x + \underbrace{C_4 x e^x - x e^x}_{\text{and for this particular example, can group once again as } (C_4 - 1)x e^x} + x e^x \ln|x|$$

$$= C_3 e^x + C_5 x e^x + x e^x \ln|x|$$

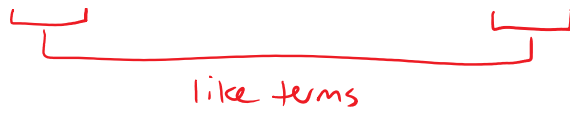
in general: if we find

$$u_1 = \int \frac{w_1}{w} dx = R(x) + C$$

$$u_2 = \int \frac{w_2}{w} dx = T(x) + D$$

then $y_p = u_1 y_1 + u_2 y_2$

$$y = y_c + y_p = C_1 y_1 + C_2 y_2 + [R(x) + C] y_1 + [T(x) + D] y_2$$



like terms

and these terms can combine together