# Explainer: Why no +C in Variation of Parameters

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consider the DE

$$y'' - 2y' + y = \frac{e^x}{x}$$

$y_c$ :
$$y'' - 2y' + y = 0$$
$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$

$$y_c = (C_1 + C_2 x) e^x$$

$$y_1 = e^x, \qquad y_2 = xe^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & xe^x + e^x \end{vmatrix} = -e^{2x}$$

$$W_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ e^x & e^x/x \end{vmatrix} = \frac{1}{x} e^{2x}$$

$$u_1 = \int \frac{W_1}{W}\, dx = \int \frac{-e^{2x}}{e^{2x}}\, dx = \int -dx = -x + C$$

$$u_2 = \int \frac{W_2}{W}\, dx = \int \frac{\frac{1}{x} e^{2x}}{e^{2x}}\, dx = \int \frac{1}{x}\, dx = \ln|x| + 0$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= (-x + C) e^x + (\ln|x| + 0) xe^x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= (-x + C)e^x + (\ln|x| + D)xe^x$$

$$y = y_c + y_p$$

$$= (C_1 + C_2 x)e^x + (-x + C)e^x + (\ln|x| + D)xe^x$$

$$= C_1 e^x + C_2 x e^x - xe^x + Ce^x + xe^x \ln|x| + Dxe^x$$

$$= \underbrace{(C_1 + C)}_{\text{new constant } C_3} e^x + \underbrace{(C_2 + D)}_{\text{new constant } C_4} xe^x - xe^x + xe^x \ln|x|$$

$$= C_3 e^x + \underbrace{C_4 xe^x - xe^x}_{\substack{\text{and for this particular example,} \\ \text{can group once again} \\ \text{as } (C_4 - 1)xe^x}} + xe^x \ln|x|$$

$$= C_3 e^x + C_5 xe^x + xe^x \ln|x|$$

in general: if we find $u_1 = \int \frac{W_1}{W} dx = R(x) + C$

$u_2 = \int \frac{W_2}{W} dx = T(x) + D$

then $y_p = u_1 y_1 + u_2 y_2$

$$y = y_c + y_p = C_1 y_1 + C_2 y_2 + [R(x) + C] y_1 + [T(x) + D] y_2$$

like terms

and these terms can combine together