# Explainer: Cauchy-Euler DEs

Tuesday, February 18, 2020 9:19 AM

Consider the DE

$$x^2 y'' + bxy' + cy = g(x)$$

Note that in standard form

$$y'' + \frac{b}{x} y' + \frac{c}{x^2} y = \frac{g(x)}{x^2}$$

and our interval of solution cannot contain the point $x = 0$. If we take our interval of solution to be $x > 0$, and substitute

$$x = e^t$$

$$\ln x = t$$

then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{1}{x}\right) \\
&= \frac{d}{dx}\left(\frac{dy}{dt}\right) \cdot \frac{1}{x} + \frac{dy}{dt} \cdot \left(\frac{-1}{x^2}\right) \\
&= \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} \cdot \frac{1}{x} - \frac{1}{x^2}\frac{dy}{dt} \\
&= \frac{d^2y}{dt^2} \cdot \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2}\frac{dy}{dt} \\
&= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)
\end{aligned}$$

substituting back into DE:

$$x^2 y'' + bxy' + cy = g(x)$$

$$x^2 \cdot \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + bx \cdot \frac{1}{x}\frac{dy}{dt} + cy = g(e^t)$$

$$\frac{d^2y}{dt^2} + (b-1)\frac{dy}{dt} + cy = g(e^t)$$

which is now a $2^{nd}$ order linear DE with constant coefficients. Solving for complementary solution gives

aux eqn: $m^2 + (b-1)m + c = 0$

$$m^2 - m + bm + c = 0$$

$$\underbrace{m(m-1) + bm + c = 0}_{\text{solve to get } m_1 \text{ and } m_2}$$

will give solutions to the homogeneous DE

① 2 distinct real: $m_1$ and $m_2$

$$y_c = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

$$= C_1 e^{m_1 \ln x} + C_2 e^{m_2 \ln x}$$

$$= C_1 e^{\ln x^{m_1}} + C_2 e^{\ln x^{m_2}}$$

$$= C_1 x^{m_1} + C_2 x^{m_2}$$

② 1 repeated: $m$

$$y_c = (C_1 + C_2 t)e^{mt}$$

$$= (C_1 + C_2 \ln x)e^{m \ln x}$$

$$= (c_1 + c_2 \ln x) e^{m \ln x}$$

$$= (c_1 + c_2 \ln x) x^m$$

3 2 complex

$\alpha \pm \beta i$

$$y_c = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

$$= e^{\alpha \ln x}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)$$

$$= x^m [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$