Math 252: Quick Review of Power Series

Power Series:

A power series in $(x-a)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

For an infinitely differentiable function, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Interval of Convergence:

For convergence of a power series, three cases:

1. The series converges for all x, so $R = \infty$.
   - For example, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$.
2. The series converges only when $x = a$ (the degenerate case), so $R = 0$.
   - For example, $\sum_{n=0}^{\infty}(n!)x^n = 1 + x + 2!x^2 + 3!x^3 + ...$ diverges for all $x \neq 0$.
3. The series converges for all x in the interval $(a-R, a+R)$ where R is the radius of convergence.
   - For example, $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$ for $-1 < x < 1$.

Math 252: Definitions for Series Solutions to DEs

Functions Which Are Analytic at a Point

A function $f(x)$ is analytic at a point a if it can be expressed as a power series centred at a with either a positive radius of convergence $R > 0$ or an infinite radius of convergence $R = \infty$.

Examples:

- $f(x) = \sin x$ is analytic at zero because the power series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

  has an infinite R.
- $f(x) = \ln x$ is not analytic about $x = 0$ because $f'(x) = \frac{1}{x}$ and so $f'(0)$ does not exist.
- $f(x) = \frac{P(x)}{Q(x)}$ is not analytic at any x for which $Q(x)$ is zero.

Ordinary vs. Singular Points:

Consider the following DE in standard form:

$$y'' + P(x)y' + Q(x)y = 0$$

A point x_0 is an ordinary point of this DE if both $P(x)$ and $Q(x)$ are analytic at x_0. Otherwise x_0 is a singular point.

Examples:

- $y'' + xy = 0$ has no finite singular points. All finite values of x are ordinary points.
- $x^2y'' + xy' - y = 0$ (Cauchy-Euler) has a singular point at $x = 0$.
- $(x^2 + 1)y'' - y = 0$ has singular points at $\pm i$. (Singular points can be imaginary.)
- $y'' - y' \ln x + y \sin x = 0$ has a singular point at $x = 0$ due to $\ln x$ term.

In the case that $P(x)$ and $Q(x)$ are quotients of polynomials, any value of x that makes the denominator zero will be a singular point.

In this course, we are only covering series solutions about ordinary points.