Math 252 – Homogeneous Linear Systems

To solve a system of first-order linear DEs, first find the eigenvalues for the coefficient matrix of the system. For a 2×2 system, you will necessarily get a quadratic in λ. The form of the solution to the system of DEs will depend on the form of the solutions to the quadratic.

Two Distinct Real Eigenvalues

If there are two distinct real eigenvalues λ_1 and λ_2, with associated eigenvectors $\mathbf{K_1}$ and $\mathbf{K_2}$, respectively, then the general solution to the DE will be

$$\mathbf{X} = c_1\mathbf{K_1}e^{\lambda_1 t} + c_2\mathbf{K_2}e^{\lambda_2 t}$$

One Repeated Real Eigenvalue

If there is only one repeated real eigenvalue λ_1 with associated eigenvector $\mathbf{K_1}$, then the two solution vectors to the DE will be

$$\mathbf{X}_1 = \mathbf{K_1}e^{\lambda_1 t} \qquad \text{and}$$

$$\mathbf{X_2} = \mathbf{K_1}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t} \qquad \text{where } \left(\mathbf{A} - \lambda_1\mathbf{I}\right)\mathbf{P} = \mathbf{K_1}$$

and the general solution to the DE will be

$$\begin{aligned}\mathbf{X} &= c_1\mathbf{X_1} + c_2\mathbf{X_2} \\ &= c_1\mathbf{K_1}e^{\lambda_1 t} + c_2\left(\mathbf{K_1}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t}\right).\end{aligned}$$

Two Complex Eigenvalues

If there are two complex eigenvalues $\lambda = \alpha \pm \beta i$, such that $\lambda_1 = \alpha + \beta i$ has associated eigenvector $\mathbf{K_1}$, then the eigenvector $\mathbf{K_2}$ associated with $\lambda_2 = \alpha - \beta i$ will be the complex conjugate of $\mathbf{K_1}$. If we combine these, letting $\mathbf{B_1}$ be the real part and $\mathbf{B_2}$ the imaginary part of $\mathbf{K_1}$ respectively, we can get two linearly independent solutions

$$\mathbf{X}_1 = \left(\mathbf{B_1}\cos\beta t - \mathbf{B_2}\sin\beta t\right)e^{\alpha t} \text{ and}$$

$$\mathbf{X}_2 = \left(\mathbf{B_1}\sin\beta t + \mathbf{B_2}\cos\beta t\right)e^{\alpha t}.$$

The general solution to the DE will be $\mathbf{X} = c_1\mathbf{X_1} + c_2\mathbf{X_2}$.

Last modified on June 13, 2011 by Patricia Wrean.