

Review Questions for Final

Wednesday, April 12, 2023 12:00 PM

solve : $x \frac{dy}{dx} + 3y = 4e^{2x}$, $y(1) = 0$

→ standard form

$$\frac{dy}{dx} + \underbrace{\left(\frac{3}{x}\right)}_{=P(x)} y = \frac{4}{x} e^{2x}$$

$$\begin{aligned} IF &= e^{\int P(x) dx} = e^{\int \frac{3}{x} dx} \\ &= e^{3 \ln x} \\ &= e^{\ln x^3} = \boxed{x^3} \end{aligned}$$

$$x^3 \frac{dy}{dx} + 3x^2 y = 4x^2 e^{2x}$$

$$\frac{d}{dx} [y x^3] = 4x^2 e^{2x}$$

$$y x^3 = \int 4x^2 e^{2x} dx$$

etc

- initial condition

solve $(y \cos x + \overset{M}{3x^2 e^y}) dx + (\sin x + \overset{N}{x^3 e^y} + 3) dy = 0$

$$\frac{\partial M}{\partial y} = \cos x + 3x^2 e^y$$

$$\frac{\partial N}{\partial x} = \cos x + 3x^2 e^y$$

} equal so DE is exact

there is an $f(x, y)$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

$$f(x,y) = \int M dx$$

$$= \int (y \cos x + 3x^2 e^y) dx$$

$$= y \sin x + x^3 e^y + g(y)$$

$$\text{and } f(x,y) = \int N dy$$

$$= \int (\sin x + x^3 e^y + 3) dy$$

$$= y \sin x + x^3 e^y + 3y + h(x)$$

$$h(x) = 0$$

$$f(x,y) = y \sin x + x^3 e^y + 3y = C$$

example: when a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in meters).

In clear seawater, the intensity one meter below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam five meters below the surface?

$$\frac{dI}{dt} \propto I$$

$$\frac{dI}{dt} = kI$$

← separable & linear 1st order

$$\int \frac{dI}{I} = \int k dt$$

$$\ln |I| = kt + C$$

$$|I| = e^{kt + C}$$

$$I = \pm e^C e^{kt}$$

$$I = C_1 e^{kt}$$

at $t=0$, $I=I_0$

$$I_0 = C_1 e^{k \cdot 0} \quad \text{so} \quad C_1 = I_0$$

$$I = I_0 e^{kt}$$

at $t=1\text{m}$, $I = 0.25 I_0$

$$0.25 \cancel{I_0} = \cancel{I_0} e^{k \cdot 1}$$

$$0.25 = e^k$$

$$\ln 0.25 = k$$

at $t=5\text{m}$, $I=?$

$$\begin{aligned} I &= I_0 e^{kt} \\ &= I_0 e^{(\ln 0.25)5} \\ &= I_0 e^{\ln (0.25)^5} \\ &= (0.25)^5 I_0 \end{aligned}$$

$$\begin{aligned} I &\approx 0.00097656 I_0 \\ I &\approx 0.00098 I_0 \end{aligned}$$

so intensity is 0.098% of original at 5m

solve the DE

$$x y' - y = 2y^3$$

std form:

$$y' - \frac{y}{x} = \frac{2y^3}{x}$$

Bernoulli:
with $n=3$

$$\text{let } u = y^{1-n} = y^{-2}$$

du = -2y⁻³ dy

$$\text{let } u = y = y$$

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx}$$

multiply DE by this

$$-2y^{-3} \frac{dy}{dx} + \frac{2y^{-2}}{x} = \frac{-4}{x}$$

$$\frac{du}{dx} + \frac{2}{x} u = \frac{-4}{x} \quad \text{linear 1st order in } u$$

$$IF = e^{\int P(x) dx}$$

$$= e^{\int \frac{2}{x} dx}$$

$$= e^{2 \ln x} = e^{\ln x^2}$$

$$= x^2$$

$$x^2 \frac{du}{dx} + 2xu = -4x$$

$$\frac{d}{dx}(ux^2) = -4x$$

$$ux^2 = -2x^2 + C$$

$$y^{-2} x^2 = -2x^2 + C$$

example: solve

$$\frac{dy}{dx} = 1 + e^{y-x+5}$$

$f(Ax+By+C)$

$$\frac{du}{dx} + 1 = 1 + e^u$$

$$\text{let } u = y - x + 5$$

$$\frac{du}{dx} = \frac{dy}{dx} - 1$$

$$\frac{dv}{dx} + 1 = 1 + e^u$$

$$\frac{dv}{dx} = \frac{dy}{dx} - 1$$

$$\frac{dv}{dx} = e^u$$

$$\frac{dv}{e^u} = dx$$

$$e^{-u} dv = dx$$

$$-e^{-u} = x + C$$

$$-e^{-y+x-5} = x + C$$

note: other examples of DEs that use this method:

$$\frac{dy}{dx} = \sin^2(2x+y+3) \quad (\text{ish})$$

$$\frac{dy}{dx} = (y+x)^2$$

example: solve

$$\frac{dy}{dx} = \frac{y^3 - x^3}{xy^2}$$

all degree 3

homogeneous

$$\text{let } y = vx$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

sub into DE:

$$v + x \frac{dv}{dx} = \frac{(vx)^3 - x^3}{x^2 v^2} = \frac{v^3 x^3 - x^3}{x^2 v^2} = \frac{v^3 - 1}{v^2}$$

$$v + x \frac{dv}{dx} = \frac{(vx)^3 - x^3}{x(vx)^2} = \frac{v^3 x^3 - x^3}{v^2 x^3} = \frac{v^3 - 1}{v^2}$$

$$v + x \frac{dv}{dx} = v - \frac{1}{v^2}$$

$$x \frac{dv}{dx} = -\frac{1}{v^2}$$

$$\int v^2 dv = \int -\frac{1}{x} dx$$

$$\frac{v^3}{3} = -\ln|x| + C$$

but $y = vx$
 $v = \frac{y}{x}$

$$\boxed{\frac{1}{3} \frac{y^3}{x^3} = -\ln|x| + C}$$

second order DEs:

example: solve $y'' - y' - 2y = 2x + 5e^{2x}$

y_c : aux eqn

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

$$y_c = c_1 e^{2x} + c_2 e^{-x}$$

y_p :

$$RHS = 2x + 5e^{2x}$$

$$y_p = Ax + B + \cancel{C}e^{2x}$$

bad case

$$y_p = Ax + B + Cxe^{2x}$$

$$y_p' = A + \underset{,}{C}e^{2x} + \underset{,}{2C}xe^{2x}$$

$$y_p' = A + Ce^{2x} + 2Cxe^{2x}$$

$$y_p'' = 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} = 4Ce^{2x} + 4Cxe^{2x}$$

plus back into DE:

$$y'' - y' - 2y = 2x + 5e^{2x}$$

$$[4Ce^{2x} + 4Cxe^{2x}] - [A + Ce^{2x} + 2Cxe^{2x}] - 2[Ax + B + Cxe^{2x}]$$

$$3Ce^{2x} - A - 2Ax - 2B = 2x + 5e^{2x}$$

$$3C = 5$$

$$\therefore C = 5/3$$

$$-2A = 2$$

$$A = -1$$

$$-A - 2B = 0$$

$$B = 1/2$$

$$y_p = -x + 1/2 + 5/3 xe^{2x}$$

general solution:

$$y = y_c + y_p$$

$$y = C_1 e^{2x} + C_2 e^{-x} - x + 1/2 + 5/3 xe^{2x}$$

example: use variation of parameters to solve

$$x^2 y'' - 5xy' + 5y = 6x^2, \quad y(1) = 3, \quad y'(1) = 9$$

Cauchy-Euler →

y_c :

aux eqn:

$$m(m-1) - 5m + 5 = 0$$

$$m^2 - 6m + 5 = 0$$

$$(m-5)(m-1) = 0$$

$$m = 5, 1$$

$$y_c = C_1 x^5 + C_2 x$$

y_p : std form: $y'' - \frac{5}{x} y' + \frac{5}{x^2} y = \boxed{6}$ $\nwarrow f(x)$
 $y_1 = x^5$
 $y_2 = x$
 $f(x) = 6$

\nearrow note:
not $6x^2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^5 & x \\ 5x^4 & 1 \end{vmatrix} = x^5 - 5x^5 = -4x^5$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x \\ 6 & 1 \end{vmatrix} = -6x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^5 & 0 \\ 5x^4 & 6 \end{vmatrix} = 6x^5$$

$$\text{then } v_1 = \int \frac{W_1}{W} dx = \int \frac{-6x}{-4x^5} dx = \frac{3}{2} \int x^{-4} dx = -\frac{1}{2} x^{-3}$$

$$v_2 = \int \frac{W_2}{W} dx = \int \frac{6x^5}{-4x^5} dx = -\frac{3}{2} \int dx = -\frac{3}{2} x$$

$$\begin{aligned}
 y_p &= v_1 y_1 + v_2 y_2 \\
 &= -\frac{1}{2} x^{-3} x^5 - \frac{3}{2} x \cdot x = -2x^2
 \end{aligned}$$

finally:

$$y = y_c + y_p$$

$$\boxed{y = c_1 x^5 + c_2 x - 2x^2}$$

IVP: $y(1) = 3$

$$3 = c_1 + c_2 - 2$$

$y'(1) = 9$

$$\begin{aligned}
 y' = 9 &= 5c_1 x^4 + c_2 - 4x \\
 9 &= 5c_1 + c_2 - 4
 \end{aligned}$$

$$\begin{aligned}
 \text{system: } \begin{cases} c_1 + c_2 = 5 \\ 5c_1 + c_2 = 13 \end{cases} &\Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 3 \end{aligned}
 \end{aligned}$$

$$y = 2x^5 + 3x - 2x^2$$

example: solve the following DE, given that $y_1 = e^{-2x^2}$ is a solution.

reduction
of
order

$$y'' + 8xy' + (4 + 16x^2)y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int p(x) dx}}{y_1^2} dx$$

← on formula sheet

$$= e^{-2x^2} \int \frac{e^{-\int 8x dx}}{(e^{-2x^2})^2} dx$$

$$= e^{-2x^2} \int \frac{e^{-4x^2}}{e^{-4x^2}} dx$$

$$= e^{-2x^2} \int dx$$

$$= x e^{-2x^2}$$

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{-2x^2} + C_2 x e^{-2x^2}$$

2nd order word problems: mass-spring systems:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}(t)$$

↑

↑

spring

↑
damping
constant

↑
spring
constant

if no damping, may rewrite as $x(t) = A \sin(\omega t + \phi)$

if damping, is it
critically damped ?
over " ?
under " ?