Section 4.2: Reduction of Order

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standard form for a homogeneous 2^{nd} order linear DE:

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

given one solution y_1,
you can find a second linearly independent solution:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx$$

then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

example: consider the DE $y'' = 6y' - 9y$
with solution $y_1 = e^{3x}$

a) show that y_1 is indeed a solution to the DE

b) find a second linearly independent solution

c) give the general solution

answer: a) $y_1 = e^{3x}$
$y_1' = 3e^{3x}$
$y_1'' = 9e^{3x}$

plug into DE: $y'' = 6y' - 9y$

$$9e^{3x} = 6(3e^{3x}) - 9(e^{3x})$$
$$9e^{3x} = 18e^{3x} - 9e^{3x} \quad \checkmark$$

b) $y'' = 6y' - 9y$

rewrite in standard form:

$$y'' - 6y' + 9y = 0$$

std form $\quad y'' + P(x)\,y' + Q(x)\,y = 0$

$$P(x) = -6$$
$$\int P(x)\,dx = \int -6\,dx = -6x$$

← once again, don't bother with the $+C$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$
$$= e^{3x} \int \frac{e^{6x}}{(e^{3x})^2}\,dx$$
$$= e^{3x} \int \frac{e^{6x}}{e^{6x}}\,dx$$
$$= e^{3x} \int dx$$
$$\boxed{y_2 = xe^{3x}}$$

← once again, omit the $+C$

c) $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 e^{3x} + c_2 x e^{3x}}$$

example: find a second linearly independent solution to the following DE on the interval $(0, \infty)$. Then state the general solution

already in std form $\rightarrow$ $$y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0$$

with solution $y_1 = x^2$

answer: $$P(x) = \frac{2}{x}$$

$$\int P(x)dx = \int \frac{2}{x} dx = 2 \ln|x|$$

note: if you like, can drop | | signs because interval of solution is $(0, \infty)$

$$e^{-\int P(x)dx} = e^{-2\ln x}$$
$$= e^{\ln x^{-2}}$$
$$= \frac{1}{x^2} = x^{-2}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= x^2 \int \frac{x^{-2}}{(x^2)^2} dx$$

$$= x^2 \int x^{-6} dx$$

$$= x^2 \left(-\frac{1}{5} x^{-5} \right)$$

$$= -\frac{1}{5} x^{-3}$$ or just $y_2 = x^{-3}$

$= -\frac{1}{5} x^{-5}$ or just $y_2 = x^{-3}$

general solution: $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 x^2 + c_2 x^{-3}}$$