Section 4.3: Homogeneous Linear DEs with Constant Coefficients

Friday, January 31, 2020 9:47 AM

example: solve $y'' - 4y' - 21y = 0$

method: write the auxiliary equation:

$$m^2 - 4m - 21 = 0$$

$$(m-7)(m+3) = 0$$

$$m = 7, -3$$

roots	Solutions
distinct real numbers $m_1, m_2\ m_3 \dots$	$e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \dots$
repeated real numbers $m_1, m_1, m_1, \dots$	$e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}, \dots$
complex numbers $\alpha \pm \beta i$	$e^{\alpha x}\cos \beta x,\ e^{\alpha x}\sin \beta x$

so for our example, $y'' - 4y' - 21y = 0$

$$m = 7, -3$$

two distinct real roots

$y_1 = e^{7x} \qquad y_2 = e^{-3x}$

general solution $\quad y = c_1 e^{7x} + c_2 e^{-3x}$

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example: solve $\quad 2y'' - 24y' + 72y = 0$

aux eqn: $\quad 2m^2 - 24m + 72 = 0$

$$2(m^2 - 12m + 36) = 0$$

$$2(m-6)^2 = 0$$

$$m = 6, 6$$

$$y = c_1 e^{6x} + c_2 x e^{6x}$$

$$= (c_1 + c_2 x) e^{6x}$$

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example: solve $\quad y'' - 6y' + 13y = 0$

aux eqn: $\quad m^2 - 6m + 13 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 13}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i$$

but $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ for $m = \alpha \pm \beta i$

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

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example: solve $y'' + 49y = 0$, $y(0) = -5$, $y'(0) = 21$

aux eqn: $m^2 + 49 = 0$

$m = \pm 7i$ $\qquad \alpha \pm \beta i$, $\alpha = 0$

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$y = \cancel{e^{0}}^{1}(c_1 \cos 7x + c_2 \sin 7x)$$

$$y = c_1 \cos 7x + c_2 \sin 7x$$

but at $x = 0$, $y = -5$

$$-5 = c_1 \cancel{\cos 0}^{1} + c_2 \cancel{\sin 0}^{0}$$

$-5 = c_1$

$y = -5\cos 7x + c_2 \sin 7x$

$y' = 35 \sin 7x + 7c_2 \cos 7x$

at $x = 0$, $y' = 21$

$21 = 35 \cancel{\sin}^{0} 0 + 7c_2 \cancel{\cos}^{1} 0$

$c_2 = 3$

$$\boxed{y = -5 \cos 7x + 3 \sin 7x}$$

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solve the following DEs:

a) $y'' + y' = 0$
b) $y'' + y = 0$
c) $y'' - y = 0$
d) $y'' = 0$

a) aux eqn: $m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

$y = c_1 + c_2 e^{-x}$

b) aux: $m^2 + 1 = 0$

$m = 0 \pm i$

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$y = c_1 \cos x + c_2 \sin x$

c) aux: $m^2 - 1 = 0$

d) aux: $m^2 = 0$

c) aux: $m^2 - 1 = 0$

$m = \pm 1$

$y = c_1 e^{x} + c_2 e^{-x}$

d) aux: $m^2 = 0$

$m = 0, 0$

$y = (c_1 + c_2 x) e^{mx}$

$y = c_1 + c_2 x$

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example: solve $y^{(4)} + 8y'' + 16y = 0$

aux eqn: $m^4 + 8m^2 + 16 = 0$

$(m^2 + 4)(m^2 + 4) = 0$

$m = \pm 2i, \pm 2i$

so $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

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in this course, we will skip any higher order DEs where the factoring of the auxiliary equation is not immediately obvious

so: $m^4 - 8m^2 + 16$ is okay

$2m^3 - 7m^2 - 7m + 2$ is not