Section 4.3: Homogeneous Linear DEs with Constant Coefficients

Thursday, February 02, 2023 2:31 PM

example: Solve $y'' - 4y' - 21y = 0$

method: write the auxiliary equation:

$$m^2 - 4m - 21 = 0$$
$$(m-7)(m+3) = 0$$
$$m = 7, -3$$

roots	solutions
distinct real numbers $m_1, m_2, m_3, \dots$	$e^{m_1 x}, e^{m_2 x}, e^{m_3 x}$
repeated real numbers m_1, m_1, m_1	$e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}$
complex numbers $\alpha \pm \beta i$	$e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$

note that the interval of solution is $\mathbb{R}$ $(-\infty, \infty)$

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so for our example, $y'' - 4y' - 21y = 0$

$m = 7, -3$

two distinct real roots

$y_1 = e^{7x}$ and $y_2 = e^{-3x}$

general solution is $y = c_1 e^{7x} + c_2 e^{-3x}$

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example: solve $2y'' - 24y' + 72y = 0$

aux eqn: $2m^2 - 24m + 72 = 0$

$$2(m^2 - 12m + 36) = 0$$

$$2(m-6)^2 = 0$$

$$m = 6, 6$$

$$\left.\begin{aligned} y &= c_1 e^{6x} + c_2 x e^{6x} \\ &= (c_1 + c_2 x) e^{6x} \end{aligned}\right\} \text{either}$$

example: solve $y'' - 6y' + 13y = 0$

aux eqn: $m^2 - 6m + 13 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 13}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i$$

but $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ for $m = \alpha \pm \beta i$

$$\boxed{y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)}$$

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example: solve $y'' + 49y = 0$, $y(0) = -5$, $y'(0) = 21$

aux eqn: $m^2 + 49 = 0$

$m^2 = -49$
$m = \pm 7i$

$m = \alpha \pm \beta i$
$\uparrow$
$\alpha = 0$

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$y = \cancel{e^{0}}^{1}(c_1 \cos 7x + c_2 \sin 7x)$

$y = c_1 \cos 7x + c_2 \sin 7x$

but at $x=0$, $y = -5$

$-5 = c_1 \cancel{\cos 0}^{1} + c_2 \cancel{\sin 0}^{0}$

$-5 = c_1$

$y = -5 \cos 7x + c_2 \sin 7x$

$y' = 35 \sin 7x + 7c_2 \cos 7x$

at $x=0$, $y' = 21$

$21 = 35 \cancel{\sin 0}^{0} + 7c_2 \cancel{\cos 0}^{1}$

$c_2 = 3$

$$\boxed{y = -5 \cos 7x + 3 \sin 7x}$$

section 4.3: cont'd 2023/02/03

means the fourth deriv y''''

example: solve $y^{(4)} + 8y'' + 16y = 0$

aux eqn: $m^4 + 8m^2 + 16 = 0$

$(m^2+4)(m^2+4) = 0$

$m = \pm 2i, \pm 2i$

so $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

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in this course we will skip any higher order DEs where the factoring of the auxiliary equation is not immediately obvious

so $m^4 + 8m^2 + 16 = 0$ is okay

$2m^3 - 7m^2 - 7m + 2 = 0$ is not

Solve $y^{(8)} - y^{(4)} = 0$

aux eqn: $m^8 - m^4 = 0$

$m^4(m^4 - 1) = 0$

$m^4(m^2 - 1)(m^2 + 1) = 0$

$m = 0, 0, 0, 0, 1, -1, 0 \pm i$

so $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x}$
$+ c_7 \cos x + c_8 \sin x$

recall for complex m: $m = \alpha \pm \beta i$

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$