Section 4.4: Non homogeneous Linear DEs with Constant Coefficients

Friday, February 03, 2023 11:07 AM

we use the "method of undetermined coefficients"

example: solve $y'' - 5y' + 6y = 5x$

RHS $\neq 0$ makes this nonhomogeneous

step ①: solve the associated homogeneous DE

$$y'' - 5y + 6y = 0$$

aux eqn: $m^2 - 5m + 6 = 0$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

"complementary solution"

step ②: find y_p, the particular solution

the RHS is $g(x) = 5x$

y_p consists of all terms in $g(x)$, $g'(x)$, $g''(x)$, ...

$$g(x) = 5x$$
$$g'(x) = 5$$
$$g''(x) = 0$$

$$y_p = Ax + B$$

step ③: put y_p back into DE to determine the value of A and B

Section 4.4: cont'd
2023/02/08

$$y_p = Ax + B$$

$$y_p' = A$$

$$y_p'' = 0$$

DE: $y'' - 5y' + 6y = 5x$

$$0 - 5A + 6(Ax + B) = 5x$$

$$6Ax + 6B - 5A = 5x + 0$$

and match coefficients:

$$0 - 5A + 6(Ax+B) = 5x$$

$$6Ax + 6B - 5A = 5x + 0$$

and match coefficients:

$$6A = 5$$
$$6B - 5A = 0$$

so $A = \frac{5}{6}$, $B = \frac{25}{36}$

so $y_p = \underbrace{\frac{5}{6}x + \frac{25}{36}}_{\text{all constant are known} \therefore \text{particular}}$

step (4): combine y_c and y_p to get y

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{3x} + \frac{5}{6}x + \frac{25}{36}}$$

---

methodology:

so if you have

$$ay'' + by' + cy = g(x)$$

where a, b, and c are constants

then

$$y = y_c + y_p$$

complementary solution to homogeneous $ay'' + by' + cy = 0$ (Section 4.3)

particular solution – y_p contains no arbitrary constants (no c_1 or c_2)

how do we find y_p?

y_p will contain all possible terms of $g(x)$ and its derivatives

$g(x)$ RHS of DE	y_p

$g(x)$ RHS of DE	y_p
polynomial of degree n	$y_p = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$
eg: $g(x) = 3x^2 + 4$	$y_p = Ax^2 + Bx + C$
sines and cosines	
eg: $g(x) = 3\sin 2x$	$y_p = A\sin 2x + B\cos 2x$
exponentials:	
eg: $g(x) = 5e^{-2x}$	$y_p = Ae^{-2x}$
eg: $g(x) = x^2 e^{-2x}$	$y_p = Ax^2 e^{-2x} + Bxe^{-2x} + Ce^{-2x}$

note: the interval of solution is still $\mathbb{R}$ $(-\infty, \infty)$ because all of these functions are well-behaved everywhere

so how do we determine the constants in y_p?

- we plug y_p back into the DE and then match coefficients

---

example: State the form of the particular solution to the following DEs. Leave your answer with undetermined coefficients.

don't bother to solve for $A, B, C,$ etc

a) $y'' - 5y' + 6y = 4x^2 - 3x$

$y_p = Ax^2 + Bx + C$

b) $y'' - 5y' + 6y = xe^{-7x}$

$y_p = Axe^{-7x} + Be^{-7x}$

c) $y - 5y' + 6y = 2\sin x - \cos x$

$y_p = A\sin x + B\cos x$

d) $y'' - 5y' + 6y = e^x \cos x$

$y_p = Ae^x\cos x + Be^x \sin x$

e) $y'' - 5y' + 6y = 4x - 3\cos 2x$

$$y_p = \underbrace{Ax + B}_{\text{from } 4x \text{ term}} + \underbrace{C\cos 2x + D\sin 2x}_{\text{from } -3\cos 2x \text{ term}}$$

---

full example: solve $y'' - 6y' + 9y = 3\cos 2x$

step ①: solve $y'' - 6y' + 9y = 0$

aux eqn: $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3, 3$

$$y_c = (C_1 + C_2 x)e^{3x}$$

step ② RHS: $g(x) = 3\cos 2x$

$$y_p = A\cos 2x + B\sin 2x$$

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

step ③ sub back into DE:

$$y'' - 6y' + 9y = 3\cos 2x$$

$$(-4A\cos 2x - 4B\sin 2x) - 6(-2A\sin 2x + 2B\cos 2x) + 9(A\cos 2x + B\sin 2x) = 3\cos 2x$$

$$(-4A - 12B + 9A)\cos 2x + (-4B + 12A + 9B)\sin 2x = 3\cos 2x$$

Section 4.4: cont'd 2023/02/09

$$(5A - 12B)\cos 2x + (12A + 5B)\sin 2x = 3\cos 2x$$

no sin term

$5A - 12B = 3$ ⇐ mult by 5

$12A + 5B = 0$ ⇐ mult by 12

$25A - 60B = 15$

$144A + 60B = 0$

$169A = 15$

$A = \frac{15}{169}$, $B = \frac{-36}{169}$

$$y_p = A\cos 2x + B\sin 2x$$

$$= \frac{15}{169}\cos 2x - \frac{36}{169}\sin 2x$$

step ④ $y = y_c + y_p$

$$= (c_1 + c_2 x)e^{3x} + \frac{15}{169}\cos 2x - \frac{36}{169}\sin 2x$$

---

the "bad" case:

example: solve $y'' + 9y = 4\sin 3x$

answer: y_c: $m^2 + 9 = 0$

$$m = \pm 3i$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

y_p: first guess: RHS = $4\sin 3x$

$$y_p = A\sin 3x + B\cos 3x$$

these terms are "like terms" to y_c

if you substitute this y_p into the DE, you'll get LHS = 0

here's what you do:

take the family of terms in y_p that have like terms in y_c and multiply by x

first guess: $y_p = \underbrace{A\sin 3x + B\cos 3x}_{\text{mult by } x}$

$$y_p = Ax\sin 3x + Bx\cos 3x$$

so now find y_p', y_p'', and sub back into DE

$$y_p = x(A\sin 3x + B\cos 3x)$$

$$y_p = x(A\sin 3x + B\cos 3x)$$

product rule

$$y_p' = (A\sin 3x + B\cos 3x) + x(3A\cos 3x - 3B\sin 3x)$$

$$y_p'' = (3A\cos 3x - 3B\sin 3x) + (3A\cos 3x - 3B\sin 3x) + x(-9A\sin 3x - 9B\cos 3x)$$

$$= (6A\cos 3x - 6B\sin 3x) - 9x(A\sin 3x + B\cos 3x)$$

now sub back into DE:

$$y'' + 9y = 4\sin 3x$$

$$6A\cos 3x - 6B\sin 3x - 9x\cancel{(A\sin 3x + B\cos 3x)} + 9x\cancel{(A\sin 3x + B\cos 3x)} = 4\sin 3x$$

$$6A\cos 3x - 6B\sin 3x = 4\sin 3x + 0\cos 3x$$

so $-6B = 4$ and $B = -\frac{2}{3}$

$6A = 0$ so $A = 0$

and $y_p = -\frac{2}{3}x\cos 3x$

general solution: $y = y_c + y_p$

$$\boxed{y = C_1\sin 3x + C_2\cos 3x - \frac{2}{3}x\cos 3x}$$

example: find the form of y_p, but leave with undetermined coefficients

(ie. don't solve for the constants)

a) $y'' - 6y' + 8y = 5e^{2x} + x^2$ bad case

$m^2 - 6m + 8 = 0$

$(m-4)(m-2) = 0$

$m = 4, 2$

$y_c = c_1 e^{2x} + c_2 e^{4x}$

$y_p = \cancel{A e^{2x}} + Bx^2 + Cx + D$

$= Axe^{2x} + \underbrace{Bx^2 + Cx + D}_{\text{leave this alone}}$

b) $y'' - 6y' + 8y = 5e^{-2x}$

$y_c =$ same

$y_p = Ae^{-2x}$

not bad case

c) $y'' - 6y' + 9y = e^{2x} + 5e^{3x}$

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3, 3$

$y_c = (c_1 + c_2 x)e^{3x}$

bad case

$y_p = Ae^{2x} + Be^{3x}$

$= Ae^{2x} + Bxe^{3x}$ still bad case

$= Ae^{2x} + Bx^2 e^{3x}$

section 4.4: cont'd 2023/02/10

d) $y'' - 6y' + 9y = xe^{3x} + \sin x$

$y_c =$ same as above

bad case

$y_p = (Ax + B)e^{3x} + C\sin x + D\cos x$

$= (Ax^2 + Bx)e^{3x} + C\sin x + D\cos x$

$= (Ax^3 + Bx^2)e^{3x} + C\sin x + D\cos x$