Section 4.6: Variation of Parameters

Friday, February 10, 2023 11:06 AM

Find a particular solution for

$$y'' + by' + cy = f(x)$$

note: standard form

[in this section, we'll start by having b and c as constants

but in section 4.7, we'll have $b(x)$ and $c(x)$]

when section 4.4 (method of undetermined coefficients) doesn't apply

recall: section 4.4 had $f(x)$ = polynomials (RHS)
= exponentials
= sines and cosines

and products of these three types

→ no logarithms, tangents, square roots, etc

here's the big idea:

find solutions y_1 and y_2 to $y'' + by' + cy = 0$

associated homogeneous DE

and then let

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

then $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$

and integrate to get u_1 and u_2

lastly $\boxed{y_p = u_1 y_1 + u_2 y_2}$

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example: solve $5y'' + 20y = 5 \csc 2x$

Section 4.6: cont'd 2023/02/14

step (1): standard form

$$y'' + 4y = \csc 2x$$

make this coefficent equal to one

step (2): find y_c

$$y'' + 4y = 0$$

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

so $y_1 = \cos 2x$
$y_2 = \sin 2x$

step (3): calculate Ws

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$W_1 = \begin{vmatrix} 0 & \\ & \end{vmatrix} = \begin{vmatrix} 0 & \sin 2x \end{vmatrix} = -\sin 2x \csc 2x = -1$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin 2x \\ \csc 2x & 2\cos 2x \end{vmatrix} = -\sin 2x \csc 2x = -1$$

$f(x) = $ RHS in standard form

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \csc 2x \end{vmatrix} = \cos 2x \csc 2x = \cot 2x$$

step 4: integrate ratios of W to get u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\, dx = \int \frac{-1}{2}\, dx = -\frac{1}{2} x$$

don't add the constant

$$u_2 = \int \frac{W_2}{W}\, dx = \int \frac{\cot 2x}{2}\, dx$$

$$= -\frac{1}{4} \ln|\csc 2x|$$

step 5: write down y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{2} x \cos 2x - \frac{1}{4} \ln|\csc 2x| \cdot \sin 2x$$

step 6: final answer

$$y = y_c + y_p$$

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x \ln|\csc 2x|}$$

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example: solve $y'' - 3y' + 2y = x^3$

could we use the method of undetermined coefficients (section 4.4)? Sure!

$$y_p = Ax^3 + Bx^2 + Cx + D$$

and you'd get a 4x4 system to solve to get the values of A, B, C, and D

note: DE is already in standard form

find y_c:

$$y'' - 3y' + 2y = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1, 2$$

$$y_1 = e^{x}$$

$$y_2 = e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{2x} \\ x^3 & 2e^{2x} \end{vmatrix} = -x^3 e^{2x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^{x} & 0 \\ e^{x} & x^3 \end{vmatrix} = x^3 e^{x}$$

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-x^3 e^{2x}}{e^{3x}}\,dx = \int -x^3 e^{-x}\,dx$$

D		I
$-x^3$	$+$	e^{-x}
$-3x^2$	$-$	$-e^{-x}$
$-6x$	$+$	e^{-x}
-6	$-$	$-e^{-x}$
0		e^{-x}

so $u_1 = (x^3 + 3x^2 + 6x + 6)e^{-x}$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{x^3 e^{x}}{e^{3x}}\,dx = \int x^3 e^{-2x}\,dx$$

D	I
x^3	e^{-2x}
$3x^2$	$-\frac{1}{2} e^{-2x}$
$6x$	$\frac{1}{4} e^{-2x}$
6	$-\frac{1}{8} e^{-2x}$
0	$\frac{1}{16} e^{-2x}$

so $u_2 = \left(-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{6}{8}x - \frac{6}{16}\right) e^{-2x}$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= (x^3 + 3x^2 + 6x + 6) e^{-x} e^{x}$$

$$+ \left(-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{3}{4}x - \frac{3}{8}\right) e^{-2x} e^{2x}$$

$$= \frac{x^3}{2} + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8}$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + \frac{x^3}{2} + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8}}$$

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note: variation of parameters gets really messy for

$$\left.\begin{aligned} y_1 &= e^{\alpha x} \cos \beta x \\ y_2 &= e^{\alpha x} \sin \beta x \end{aligned}\right\} \text{complex case}$$

<u>not recommended</u>