# Section 4.7: Cauchy-Euler DEs

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example: consider $x^2y'' - 7xy' + 15y = 0$

note that the degree of $x$ (power/exponent) equals the order of the derivative in each term

$y''$ - second order
$x^2$ - second degree

general procedure for Cauchy Euler equations

- write auxiliary equation, but

$$x^3y''' \rightarrow m(m-1)(m-2)$$
$$x^2y'' \rightarrow m(m-1)$$
$$xy' \rightarrow m$$
$$y \rightarrow 1$$

(we're basically doing the substitution that $y = x^m$)

| roots | solutions |
|---|---|
| distinct real roots $m_1, m_2, m_3, \ldots$ | $x^{m_1}, x^{m_2}, x^{m_3}, \ldots$ |
| repeated real root $m_1, m_1, m_1, \ldots$ | $x^{m_1}, x^{m_1}\ln x, x^{m}(\ln x)^2, \ldots$ |
| complex case | $x^{\alpha}[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$ |

complex case
$m = \alpha \pm \beta i$

$$x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

note: looks like the constant coeff case,
but $e^{mx} \to x^m$
$x \to \ln x$

back to our example:

$$x^2 y'' - 7xy' + 15y = 0$$

answer:
aux eqn: $m(m-1) - 7m + 15 = 0$

$$m^2 - 8m + 15 = 0$$

$$(m-3)(m-5) = 0$$

$$m = 3, 5$$

$$\boxed{y = c_1 x^3 + c_2 x^5}$$

totally optional check

$$y = c_1 x^3 + c_2 x^5$$

$$y' = 3c_1 x^2 + 5c_2 x^4$$

$$y'' = 6c_1 x + 20 c_2 x^3$$

sub back into DE:

$$x^2 y'' - 7xy' + 15y = 0$$

$$x^2(6c_1 x + 20c_2 x^3) - 7x(3c_1 x^2 + 5c_2 x^4) + 15(c_1 x^3 + c_2 x^5) = 0$$

$$x^3(6c_1 - 21c_1 + 15c_1) + x^5(20c_2 - 35c_2 + 15c_2) = 0$$

$$0 = 0 \quad \checkmark$$

note: interval of solution for CE:

consider DE

$$x^2 y'' + bxy' + cy = g(x)$$

in standard form

$$y'' + \frac{b}{x} y' + \frac{c}{x^2} y = \frac{g(x)}{x^2}$$

the interval of solution must not contain $x=0$, so interval is either $x<0$ or $x>0$, and we choose $x>0$ for simplicity

- this allows us to drop absolute signs in $\ln x$, should that term be in the solution

example: solve $x^2 y'' - 3xy' + 4y = 0$ for $x>0$

where $y(1) = 6$ and $y'(1) = 3$

answer: aux eqn:

$$m(m-1) - 3m + 4 = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$y = c_1 x^2 + c_2 x^2 \ln x$$

initial conditions:

$$y(1) = 6$$

$$6 = c_1 + c_2 \cancel{\ln 1}^{0}$$

$$\underline{c_1 = 6}$$

and $y'(1) = 3$

$$y' = 2c_1 x + 2c_2 x \ln x + c_2 x$$

$$3 = 2c_1 + c_2$$

$$3 = 12 + c_2$$

$$c_2 = -9$$

$$\boxed{y = 6x^2 - 9x^2 \ln x}$$

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example: solve $x^2 y'' - xy' + 4y = 0$

aux eqn:

$$m(m-1) - m + 4 = 0$$

$$m^2 - 2m + 4 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2}$$

$$= 1 \pm i\sqrt{3}$$

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$$

$$= x\left[c_1 \cos(\sqrt{3} \ln x) + c_2 \sin(\sqrt{3} \ln x)\right]$$

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a higher order example:

solve: $x^3 y''' + x y' - y = 0$

aux eqn: $m(m-1)(m-2) + m - 1 = 0$

$$(m-1)\left[m(m-2) + 1\right] = 0$$

$$(m-1)\left[m^2 - 2m + 1\right] = 0$$

$$(m-1)^3 = 0$$

$$m = 1, 1, 1$$

$$y_c = c_1 x + c_2 x \ln x + c_3 x (\ln x)^2$$

example: solve $x^2 y'' - 6xy' + 12y = \frac{1}{x}$

on interval $(0, \infty)$.

this just mean you can drop any absolute value signs that show up.

answer: find $y_c$ first:

aux eqn for Cauchy Euler: $m(m-1) - 6m + 12 = 0$

$$m^2 - 7m + 12 = 0$$

$$(m-3)(m-4) = 0$$

$$m = 3, 4$$

$$y_c = c_1 x^3 + c_2 x^4$$

note that we cannot use the method of undetermined coefficients here

- $y_p$ is supposed to include all of the derivatives of the right hand side

any $x^{-1}, x^{-2}, x^{-3}$ will have derivatives that simply look like $x^{-1} \to x^{-2} \to x^{3}$ and does not go to zero or cycle back and look like the original

okay, so need to solve this Cauchy-Euler non-homogeneous DE with variation of parameters (section 4.6)

step 1: write the DE in standard form to get $f(x)$

so $$x^2 y'' - 6xy' + 12y = \frac{1}{x}$$

becomes

$$y'' - \frac{6}{x} y' + \frac{12}{x^2} y = \frac{1}{x^3}$$

this is $f(x)$

step ②: find $f(x)$ — RHS

find $y_1$ and $y_2$ from $y_c$

$$y_1 = x^3$$

$$y_2 = x^4$$

step ③ compute the Ws:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = x^6$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^4 \\ x^{-3} & 4x^3 \end{vmatrix} = -x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^3 & 0 \\ 3x^2 & x^{-3} \end{vmatrix} = 1$$

step ④: calculate $u_1$ and $u_2$:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-x}{x^6}\,dx = -\int x^{-5}\,dx = \frac{1}{4}x^{-4}$$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{1}{x^6}\,dx = \int x^{-6}\,dx = -\frac{x^{-5}}{5}$$

step ⑤: find $y_p$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}x^{-4}x^3 - \frac{1}{5}x^{-5}x^4$$

$$= \frac{1}{4}x^{-1} - \frac{1}{5}x^{-1} = \frac{1}{20x}$$

and finally $y = y_c + y_p$

$$\boxed{y = c_1 x^3 + c_2 x^4 + \frac{1}{20x}}$$