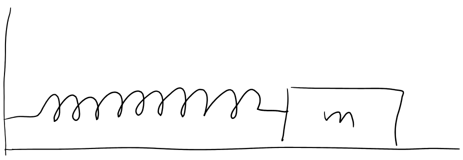
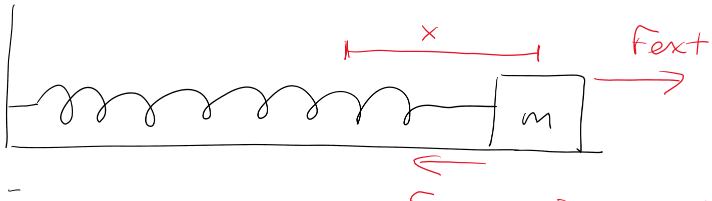
Section 5.1: Linear Models (2nd order applications)

Thursday, March 02, 2023 2:36 PM

Hooke's Law:

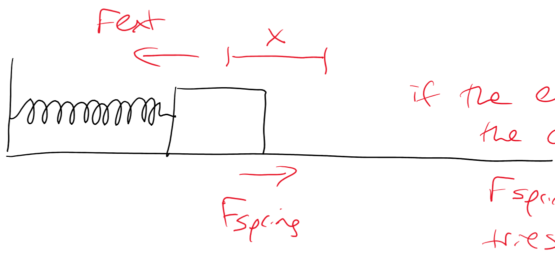


consider mass m on a frictionless surface with an ideal spring



(external force)

F_{spring} is in opposite direction, and tries to return mass to the equilibrium position



if the external force is in the opposite direction, then F_{spring} still opposes it and tries to return mass to the equilibrium position

we call this type of force a "restoring force"

Hooke's Law: $$\vec{F}_{spring} = -k\vec{x}$$

the displacement

now let's let go of the block so $F_{ext} = 0$

then

$$\sum \vec{F} = m\vec{a}$$ Newton's 2nd Law

$$\vec{F}_{spring} = m\vec{a}$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$ 2nd order linear DE

$$[\text{or} \quad m\ddot{x} + kx = 0 \,]$$

let's solve it!

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

aux eqn: use n's instead since m is now mass

$$n^2 + \frac{k}{m} = 0$$

$$n^2 = -\frac{k}{m}$$

$$n = \pm i\sqrt{\frac{k}{m}}$$

$$x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

$$= c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

a common convention is to say that $\omega_0 = \sqrt{\frac{k}{m}}$

lowercase omega

ω_0 = angular frequency

so $x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

note: ω_0 is related to f_0, the natural frequency,

by $f = \frac{\omega}{2\pi}$ and also the

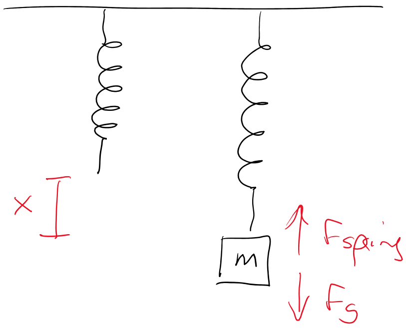
period T is given by $T = \frac{2\pi}{\omega}$

so what does this look like?

- single sine wave with a phase shift

(more details later)

so, how do you calculate k, the spring constant?



the spring stretches by x when mass m is hung from it

forces down = forces up (mass is in equilibrium)

$$F_g = F_{spring}$$

$$mg = kx$$

$$k = \frac{mg}{x}$$

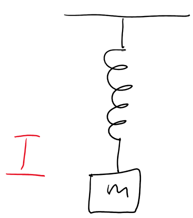
if $m = 9.0$ kg and $x = 3.0$ m, then using $g = 9.8\ m/s^2 = 9.8\ N/kg$

then $k = \frac{mg}{x} = \frac{(9.0\ kg)(9.8\ N/kg)}{3.0\ m}$

$$= 29.4\ N/m$$

$$= 29\ N/m$$

note: what happens when a spring is hung vertically and then set in motion?



$y_0 = \frac{mg}{k}$

← here is the unstretched position of the mass you get that

$$m\frac{d^2y}{dt^2} + ky = F_{ext}$$

$$m\frac{d^2y}{dt^2} + ky = mg \qquad \text{non homogeneous}$$

but we can just rewrite this as

$$m\frac{d^2y}{dt^2} + k(y-y_0) = 0$$

y_0 is the length the spring stretches when the mass is hung

and can redefine coordinates

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example: A 0.25 kg mass is hung from a spring with spring constant 1.0 N/m. The mass is released 0.50 m below equilibrium with velocity 2 m/s upwards. Assume no damping.

a) Find $y(t)$

$$\sum \vec{F} = m\vec{a}$$

$$-ky = m\frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$0.25\frac{d^2y}{dt^2} + 1.0y = 0$$

note: the force of gravity stretches the spring so that it oscillates about a new equilibrium position - so don't have to include here

aux eqn: $0.25n^2 + 1.0 = 0$

$$n^2 = \frac{-1.0}{0.25} = -4$$

$$n = \pm 2i$$

$$y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

$$y = c_1 \cos 2t + c_2 \sin 2t$$

now apply initial conditions:

↑ y (positive is up)

$$y(0) = -0.50 \text{ m}$$

$$\frac{dy}{dt}(0) = 2.0 \text{ m/s}$$

note: either convention is okay but <u>be consistent</u>

at $t = 0$, $-0.50 = c_1 \cancel{\cos 0}^{1} + c_2 \cancel{\sin 0}^{0}$

$$c_1 = -0.50$$

at $t=0$, $\frac{dy}{dt} = 2.0$

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t$$

$$2 = -2c_1 \cancel{\sin 0}^{0} + 2c_2 \cancel{\cos 0}^{1}$$

$$c_2 = 1.0$$

$$\boxed{y(t) = -0.50 \cos 2t + 1.0 \sin 2t}$$

b) write $y(t)$ in the form of $A \sin(\omega t + \phi)$

(see handout of Linear Combo of Sine and Cosine)

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \tan\phi = \frac{c_1}{c_2}$$

from our problem, $c_1 = -0.5$

$c_2 = 1.0$

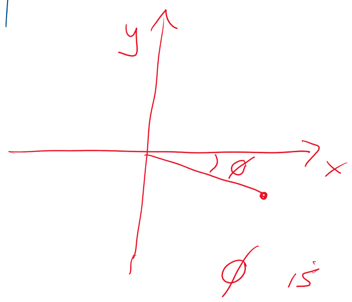
so $A = \sqrt{(-0.5)^2 + (1.0)^2}$

$= 1.11803$

$= 1.1$

$\tan\phi = \dfrac{-0.5}{1}$ $\quad \left(= \dfrac{y}{x}\right)$

$(x, y) = (1, -0.5)$



ϕ is in QIV

and $\arctan\left(\dfrac{-0.5}{1.0}\right)$

$= -26.6^\circ$

$= -0.463648$ radians

$= -0.46$ rads

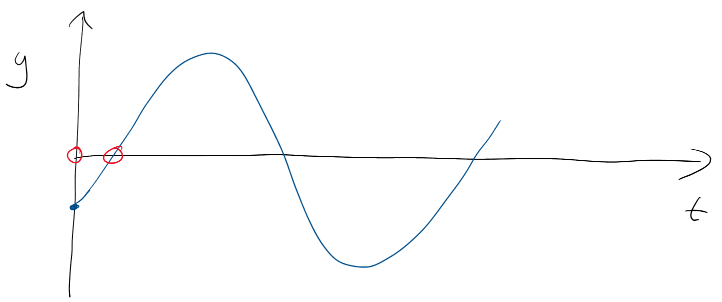
since ϕ is in QIV then $\phi = -0.46$ rads with no need to add π

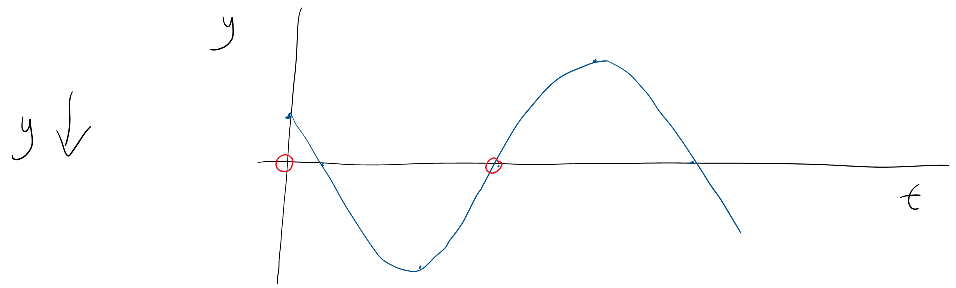
(in QII and III, will need to add π to the angle calculated from arctan function)

therefore, $y(t) = 1.1 \sin(2t - 0.46)$

note: if you use $\downarrow y$ (positive is down), you will get a different value for the phase shift

choose $y \uparrow$





c) Find the mass's maximum displacement from equilibrium position.

$\boxed{\pm 1.1 \text{ m}}$

d) Find the period of motion.

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \quad \text{seconds/cycle}$$

e) Find the time when the mass first returns to equilibrium position

$$y = 1.1 \sin(2t - 0.46)$$

$$0 = 1.1 \sin(2t - 0.46)$$

↑
infinite number of values of t that make this true — we want the smallest positive value

$$2t - 0.46 = \ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots$$

$2t - 0.46 = 0$ has a positive solution

$$2t = 0.46$$
$$t = 0.23 \text{ seconds}$$

Section S.1: cont'd 2023/03/07

so what happens when we consider air resistence?

$$F_{fr} \propto v$$

so $\vec{F}_{fr} = -b\vec{v}$

b is a positive constant

and the force of friction due to air resistence is in the opposite direction to the velocity

so then $\Sigma \vec{F} = m\vec{a}$

$$\vec{F}_{spring} + \vec{F}_{fr} = m\vec{a}$$

$$-kx - bv = ma$$

$$\boxed{\begin{aligned} &ma + bv + kx = 0 \\ &m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \\ &m\ddot{x} + b\dot{x} + kx = 0 \end{aligned}}$$

2^{nd} order linear homogeneous

aux eqn: $mn^2 + bn + k = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$

solns will be (1) 2 distinct real if $b^2 - 4km > 0$

(2) 1 repeated real if " $= 0$

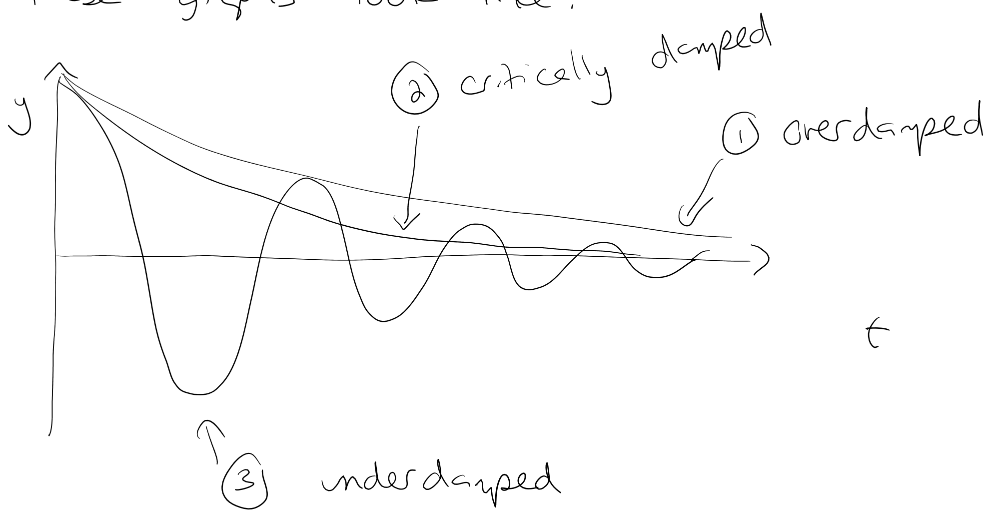
③ 2 complex if " < 0

with $y_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$y_2 = (c_1 + c_2 t) e^{rt}$

$y_3 = e^{-b/2m\, t} (c_1 \cos \beta t + c_2 \sin \beta t)$

so, what do these graphs look like?



critically damped – just enough friction to prevent oscillation

→ object returns to equilibrium in minimum time

what about an external force?

then get

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_{ext}(t)$$

example: buildings being pumped by a long-duration earthquake example: Consider the situation in which a 1 kg mass is hung from a spring with spring constant 5 N/m. Air resistance is present with a damping constant numerically equal to 2. Initially, the mass is at rest in the equilibrium position. At $t=0$, an external force $F_{ext} = \cos t$ begins to act on the system.

Find $y(t)$ and identify any transient terms in the solution.

answer:

$$m\frac{d^2y}{dt} + b\frac{dy}{dt} + ky = F_{ext}$$

$$1\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \cos t$$

y_c: aux eqn $n^2 + 2n + 5 = 0$

$n = -1 \pm 2i$

$$y_c = e^{nt}(c_1 \cos \beta t + c_2 \sin \beta t)$$

$$y_c = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

y_p: RHS $= \cos t$

$$y_p = A\cos t + B\sin t \quad \leftarrow \text{not bad case}$$

$$\frac{dy_p}{dt} = -A\sin t + B\cos t$$

$$\frac{d^2y_p}{dt^2} = -A\cos t - B\sin t$$

now sub back into DE:

$$\frac{d^2y}{dt} + 2\frac{dy}{dt} + 5y = \cos t$$

$(-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) + 5(A\cos t + B\sin t)$

$= \cos t$

$(-A + 2B + 5A)\cos t + (-B - 2A + 5B)\sin t = \cos t$

$$\left.\begin{aligned} 4A + 2B &= 1 \\ -2A + 4B &= 0 \end{aligned}\right\}$$

$$\begin{array}{r} 4A + 2B = 1 \\ -4A + 8B = 0 \\ \hline 10B = 1 \end{array}$$

$B = 1/10$

$-2A + 4B = 0$

$-2A + \frac{4}{10} = 0$

$-2A = -\frac{4}{10}$

$A = 1/5$

so $y_p = \frac{1}{5}\cos t + \frac{1}{10}\sin t$

$y = y_c + y_p$

$= e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + 1/5 \cos t + \frac{1}{10}\sin t$

initial conditions: at $t = 0$, $y = 0$ ⇐ start at equib

$\frac{dy}{dt} = 0$ ⇐ starts from rest

$y = 0 = C_1 + 0 + 1/5 + 0$

so $C_1 = -1/5$

the second constant C_2 will be much more annoying due to product rule

$$y = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + \tfrac{1}{5}\cos t + \frac{1}{10}\sin t$$

$$\frac{dy}{dt} = -e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

$$+ e^{-t}(-2c_1 \sin 2t + 2c_2 \cos 2t)$$

$$- \tfrac{1}{5}\sin t + \tfrac{1}{10}\cos t$$

at $t = 0$

$$0 = -(c_1 + 0) + (0 + 2c_2) + \tfrac{1}{10}$$

$$0 = -c_1 + c_2 + \frac{1}{10}$$

$$0 = \tfrac{1}{5} + 2c_2 + \frac{1}{10}$$

$$2c_2 = \frac{-3}{10}$$

$$c_2 = \frac{-3}{20}$$

$$\boxed{y = \underbrace{e^{-t}\left(-\frac{1}{5}\cos 2t - \frac{3}{20}\sin 2t\right)}_{\text{transient}} + \frac{1}{5}\cos t + \frac{1}{10}\sin t}$$