Section 6.2: Power Series: Solutions about Ordinary Points

Wednesday, March 08, 2023 12:00 PM

What 2^{nd} order DEs can we, in theory, solve?

$$ay'' + by' + cy = 0$$ constant coeffs

$$ax^2y'' + bxy' + cy = 0$$ Cauchy-Euler

$$y'' + P(x)y' + Q(x)y = 0$$ provided that y_1 is known (reduction of order)

otherwise, what to do?

$\boxed{\text{power series}}$

big idea: replace y by $\sum_{n=0}^{\infty} c_n (x-x_0)^n$ and find all the constants c_n (or as many of them as you want)

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see handout for definition of ordinary vs singular points

theorem: if x_0 is an ordinary point for the DE

$$y'' + P(x)y' + Q(x)y = 0$$

then we can find two LI power series solutions

$$y = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$= c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots$$

[why two series? because a 2^{nd} order DE has two arbitrary constants]

These series converge at least on some interval $|x-x_0| < R$, where R is the distance to the nearest singular point.

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example: Find two power series solutions of the given DE about the point $x=0$. Find the recurrence relation and give the first three terms in each series.

$$y'' + x^2 y = 0$$

Also, state the values of x for which these series converge. $\rightarrow$ no singular points so $x \in \mathbb{R}$

answer: let $x_0 = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

then DE becomes

$$y'' + x^2 y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

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we want to combine these two series into a single series, so reindex into series in x^k

let $k = n-2$
$k+2 = n$
when $n=2$, $k=0$

let $k = n+2$
$k-2 = n$
when $n=0$, $k=2$

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} c_{k-2} x^k = 0$$

now, write out the first two terms of the left series so that the starting index is same for both

$$2 \cdot 1 \cdot c_2 x^0 + 3 \cdot 2 c_3 x^1 + \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} c_{k-2} x^k = 0$$

so

$$2c_2 + 6c_3 x + \sum \left[(k+2)(k+1) c_{k+2} + c_{k-2} \right] x^k = 0$$

$$2c_2 + 6c_3 x + \sum_{k=2} \left[(k+2)(k+1)c_{k+2} + c_{k-2}\right] x^k = 0$$

$$= 0 + 0x + 0x^2 + 0x^3 + \ldots$$

so match coefficients - all coeffs on LHS need to equal zero

$$\boxed{\begin{aligned} 2c_2 &= 0 \\ 6c_3 &= 0 \end{aligned}}$$

and for all k from 2 to ∞

solve for the constant with the highest subscript $\longrightarrow$ $(k+2)(k+1)c_{k+2} + c_{k-2} = 0$

$$\boxed{c_{k+2} = -\frac{c_{k-2}}{(k+2)(k+1)} \quad \text{for } k = 2, 3, 4, \ldots}$$

this is called the recurrence relation

- expresses the relationship between coefficients

note: no info on c_0 or c_1

so for $k=2$, $\quad c_4 = -\dfrac{c_0}{4\cdot 3} = -\dfrac{c_0}{12}$

$k=3$, $\quad c_5 = -\dfrac{c_1}{5\cdot 4} = -\dfrac{c_1}{20}$

$k=4$, $\quad c_6 = -\dfrac{c_2}{6\cdot 5} = 0 \quad$ because $c_2 = 0$

$k=5$, $\quad c_7 = -\dfrac{c_3}{7\cdot 6} = 0 \quad c_3 = 0$

$k=6$, $\quad c_8 = -\dfrac{c_4}{8\cdot 7} = +\dfrac{c_0}{8\cdot 7\cdot 4\cdot 3}$

$k=7$, $\quad c_9 = -\dfrac{c_5}{9\cdot 8} = +\dfrac{c_1}{9\cdot 8\cdot 5\cdot 4}$

so $\quad y = c_0 + c_1 x + \cancel{c_2 x^2}^{\,0} + \cancel{c_3 x^3}^{\,0} + c_4 x^4 + \ldots$

$$= c_0 + c_1 x - \frac{c_0}{12}x^4 - \frac{c_1}{20}x^5 + \frac{c_0}{672}x^8 + \frac{c_1}{1440}x^9 + \ldots$$

now, divide into two series, one in c_0 and the other in c_1

$$\boxed{y = c_0 y_1 + c_1 y_2}$$

where $c_0 y_1 = c_0 - \frac{c_0}{12} x^4 + \frac{c_0}{672} x^8 + \ldots$

$$\boxed{y_1 = 1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 + \ldots}$$

and $c_1 y_2 = c_1 x - \frac{c_1}{20} x^5 + \frac{c_1}{1440} x^9 + \ldots$

$$\boxed{y_2 = x - \frac{1}{20} x^5 + \frac{1}{1440} x^9 + \ldots}$$

note: what's the fastest way to get y_1 and y_2?

set $c_0 = 1$ and $c_1 = 0$, result is y_1

$c_0 = 0$ and $c_1 = 1$, " y_2

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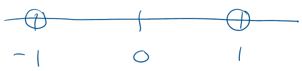
example: Find two power series solutions for the following DE about the ordinary point $x=0$. Find the recurrence relation and give the first three terms in each series. Also, state the values of x for which this series will converge.

$$(x^2-1)y'' + xy' - y = 0$$

answer: converges:

in std form $y'' + \frac{x}{x^2-1} y' - \frac{1}{x^2-1} y = 0$

singular points at $x = \pm 1$



so will converge for $\boxed{-1 < x < 1}$

so let $y = \sum_{n=0}^{\infty} c_n x^n$

$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$$y'' = \sum_{n=2} n(n-1) c_n x^{n-2}$$

now sub back into DE:

$$(x^2-1)y'' + xy' - y = 0$$

mult through

$$(x^2-1)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

let $k=n$ | let $k=n-2$, $k+2=n$, when $n=2$, $k=0$ | $k=n$ | $k=n$

$$\sum_{k=2}^{\infty} k(k-1) c_k x^k - \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

now write out terms for $k<2$

$$\sum_{k=2}^{\infty} k(k-1) c_k x^k - \left[2c_2x^0 + 3\cdot 2\cdot c_3 x^1 + \sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k\right]$$

$$+ \left[\cancel{c_1 x} + \sum_{k=2}^{\infty} k c_k x^k\right] - \left[c_0x^0 + \cancel{c_1 x} + \sum_{k=2}^{\infty} c_k x^k\right] = 0$$

clean up

$$-2c_2 - 6c_3x - c_0 + \sum_{k=2}^{\infty}\left[k(k-1)c_kx^k - (k+2)(k+1)c_{k+2}x^k + kc_kx^k - c_kx^k\right] = 0$$

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$$(-2c_2 - c_0) - 6c_3x + \sum_{k=2}^{\infty}\left[(\underbrace{k(k-1)+k-1}_{=k^2-1})c_k - (k+2)(k+1)c_{k+2}\right]x^k = 0$$

$$= 0 + 0x + 0x^2 + 0x^3 \ldots$$

set all coefficients for each term in x to zero:

$$-2c_2 - c_0 = 0$$

$$-6c_3 = 0$$

and $(k^2-1)c_k - (k+2)(k+1)c_{k+2} = 0$

write all higher coeffs in terms of lower coeffs

recurrence relation

$$\boxed{\begin{aligned} c_2 &= -\tfrac{1}{2} c_0 \\ c_3 &= 0 \\ c_{k+2} &= \frac{(k^2-1)}{(k+1)(k+2)} c_k \quad \text{for } k = 2, 3, 4, \ldots \\ &= \frac{(k-1)}{k+2} c_k \end{aligned}}$$

shortcut:

set $c_0 = 1$ and $c_1 = 0$ to get y_1

$c_2 = -\frac{1}{2} c_0 = -\frac{1}{2}$

$c_3 = 0$

for $k=2$, $c_4 = \frac{1}{4} c_2 = \frac{-1}{8}$

we have three non-zero coeffs so we can stop here

$y_1 = 1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 + \ldots$

set $c_0 = 0$ and $c_1 = 1$ to get y_2

$c_2 = -\frac{1}{2} c_0 = 0$

$c_3 = 0$

$k=2$ $c_4 = \frac{1}{4} c_2 = 0$

$c_5 = \frac{2}{5} c_3 = 0$

$c_6 = \frac{3}{7} c_5 = 0$

zeros all the way down

$y_2 = x$

$$\boxed{\begin{aligned} &\text{so} \quad y = c_0 y_1 + c_1 y_2 \\ &\text{where} \quad y_1 = 1 - \tfrac{1}{2} x - \tfrac{1}{8} x^4 + \ldots \\ &\phantom{\text{where}} \quad y_2 = x \end{aligned}}$$

note: textbook gives solutions as

$y = y_1 + y_2$

where $y_1 = c_0 \left[1 - \frac{1}{2} x - \frac{1}{8} x^4 + \ldots \right]$

$y_2 = c_1 x$

which is completely equivalent

IVPs : initial value problems

IVPs : initial value problems

suppose you have a DE with initial conditions

$$y(0) = 2$$
$$y'(0) = -3$$

how does this work?

recall: $y = c_0 + \underbrace{c_1 x + c_2 x^2 + c_3 x^3 + \dots}_{\text{when } x=0, \text{ these terms vanish}}$

$$2 = c_0$$

and

$$y' = c_1 + \underbrace{2c_2 x + 3c_3 x^2 + \dots}_{\text{when } x=0, \text{ these terms vanish}}$$

$$-3 = c_1$$

in general, $c_0 = y(0)$

$c_1 = y'(0)$

example: Give the first four terms of the power series solution to

$$y'' + xy = 0 \quad \text{for} \quad y(0) = 2$$
$$y'(0) = -3$$

note: recurrence relation is <u>not</u> required, so do instead

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$
$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$
$$y'' = 2c_2 + 3 \cdot 2\, c_3 x + 4 \cdot 3\, c_4 x^2 + \dots$$

now sub into DE:

$$y'' + xy = 0$$

$$(2c_2 + 6c_3 x + 12 c_4 x^2 + \dots) + x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + \dots) + (c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + \dots) = 0$$

now match coeffs :

$$2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + \dots = 0$$

$$= 0 + 0x + 0x^2 + \dots$$

so $2c_2 = 0$ and $c_2 = 0$

$6c_3 + c_0 = 0$ $\quad c_3 = -\tfrac{1}{6} c_0$

$12c_4 + c_1 = 0$ $\quad c_4 = -\tfrac{1}{12} c_1$

but $c_0 = y(0) = 2$ from initial conditions

$c_1 = y'(0) = -3$

$c_2 = 0$

$c_3 = -\frac{1}{6} c_0 = -\frac{1}{6}(2) = -\frac{1}{3}$

$c_4 = -\frac{1}{12} c_1 = -\frac{1}{12}(-3) = \frac{1}{4}$

finally, $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

$$\boxed{y = 2 - 3x - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots}$$

note: if you insist, could still use recurrence relation method and sub in initial values at end.

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what if your $P(x)$ or $Q(x)$ is not a polynomial?

$$y'' + e^x y' - y = 0$$

$P(x)$ not a polynomial

→ use Maclaurin or Taylor series expansion

expands around

expands around $x = a$

"↑"
expands
around
$x=0$

↑
expands around $x=a$

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quick note on multiplying series:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2$$
$$+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + \dots$$

example: find two power series solutions to the following DE about the ordinary point $x=0$.
Give the first three non-zero terms in each series.

$$y'' + e^x y' - y = 0$$

note: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

answer: let
$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$
$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$
$$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots$$

now substitute into DE:

$$y'' + e^x y' - y = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + \dots) + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)(c_1 + 2c_2 x + 3c_3 x^2 + \dots)$$
$$- (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) = 0$$

now collect like terms in x:

$$(2c_2 + c_1 - c_0) + (6c_3 + \cancel{c_1} + 2c_2 - \cancel{c_1})x$$
$$+ \left(12c_4 + 3c_3 + 2c_2 + \frac{c_1}{2} - c_2\right)x^2 + \dots = 0$$

now match coeffs

so
$$2c_2 + c_1 - c_0 = 0$$
$$6c_3 + 2c_2 = 0$$
$$12c_4 + 3c_3 + \cancel{2c_2}^{\,c_2} + \frac{c_1}{2} - \cancel{c_2} = 0$$

and solve for the coeff with the highest subscript

then
$$c_2 = \frac{c_0 - c_1}{2}$$

$$c_3 = -\frac{1}{3}c_2$$

$$c_4 = \frac{-1}{12}\left(3c_3 + c_2 + \frac{c_1}{2}\right)$$
$$= \frac{-1}{12}\left(3\left(-\tfrac{1}{3}c_2\right) + c_2 + \frac{c_1}{2}\right)$$
$$= \frac{-1}{12}\left(-c_2 + c_2 + \frac{c_1}{2}\right)$$
$$= -\frac{1}{24}c_1$$

shortcut: to find y_1
set $c_0 = 1$ and $c_1 = 0$

$c_2 = 1/2$

$c_3 = -1/6$

$c_4 = 0$ (don't need this, because already have 3 non zero terms)

to find y_2
set $c_0 = 0$ and $c_1 = 1$

$c_2 = -1/2$

$c_3 = 1/6$

$c_4 = -1/24$

finally:

$$\boxed{\begin{aligned} &y = c_0 y_1 + c_1 y_2 \\ &\text{where} \quad y_1 = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \dots \\ &\text{and} \quad y_2 = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \dots \end{aligned}}$$

note: One drawback of this approach is that if many of the coeffs are zero, you have to go back and multiply through more terms to generate the number of coeffs needed number of coeffs needed

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one last thing: what if your recurrence relation is in terms of more than one constant on the right-hand-side?

$$c_{k+2} = \frac{c_k + c_{k+1}}{(k+1)(k+2)} \qquad k = 1, 2, 3, \ldots$$

— it's exactly the same as before
so let $c_0 = 1$ and $c_1 = 0$ to get y_1,

and so on