Section 7.1: Definition of the Laplace Transform

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why do we care?

we can use the Laplace transform to reduce a linear DE to an algebraic eqn $\leftarrow$ easy to solve

then do an inverse Laplace transform to the algebraic solution to get the solution to the DE

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definition: for a function $f(t)$ which exists for $t \geq 0$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

provided that the integral converges (so s has to be large enough to make this work)

Section 7.1: cont'd 2023/03/16

examples: Find the Laplace transform of $f(t) = 1$.

answer: $$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-st}\, dt$$

$$= \lim_{b \to \infty} \left. \frac{e^{-st}}{-s} \right|_0^b$$

$$b \to \infty \qquad -s \qquad |_0$$

$$= \lim_{b \to \infty} \left(-\frac{e^{-sb}}{s} + \frac{1}{s} \right)$$

$$= \frac{1}{s}$$

similarly, $\mathcal{L}\{t\} = \dfrac{1}{s^2}$

$$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$$

for $s > -3$

need this condition for the limit to converge

now with Laplace transforms, we typically do not start from scratch with an integral, but rather consult a table:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \text{for } n = 1, 2, 3, \ldots$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad (\text{assume } s > a)$$

so for example,

$$\mathcal{L}\{t^6\} = \frac{6!}{s^7}$$

note that all transforms are now functions of s, not t

nice property: Laplace transforms are linear if $f(t) = a\,g(t) + b\,h(t)$ where a and b are constants

then $\mathcal{L}\{f(t)\} = a\,\mathcal{L}\{g(t)\} + b\,\mathcal{L}\{h(t)\}$

example: (1) $f(t) = t^2 - e^{2t} + 4$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} - \frac{1}{s-2} + \frac{4}{s}$$

(2) $f(t) = \cos 3t + \sin 2t$

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + \frac{2}{s^2+4}$$

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notation: $\mathcal{L}\{f(t)\} = F(s)$

function of t — function s

lowercase f — uppercase F

$$\mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

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how do you deal with non-linear functions?

sometimes trig identities come in handy:

(1) if $f(t) = \cos^2 t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1 + \cos 2t}{2}\right\}$$

$$= \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+4}$$

(2) if $f(t) = \cos(\omega t + \phi)$, rewrite using sum/difference

constants

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos\omega t \cos\phi - \sin\omega t \sin\phi\}$$

$$= \cos\phi \frac{s}{s^2+\omega^2} - \sin\phi \frac{\omega}{s^2+\omega^2}$$

or sometimes you just do some algebra:

(3) if $f(t) = (2t-1)^3$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{8t^3 - 12t^2 + 6t - 1\}$$

$$= 8 \cdot \frac{3!}{s^4} - 12\frac{2}{s^3} + \frac{6}{s^2} - \frac{1}{s}$$

note: (we do not test) the only place you use the definition is for piecewise functions

$$f(t) = \begin{cases} 2t+1 & 0 \le t < 1 \\ 4 & t \ge 1 \end{cases}$$

then $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(2t+1)\,dt + \int_1^\infty e^{-st}\,4\,dt$

etc

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but how do they actually work?

suppose we have

$$\boxed{\frac{dn}{dt} = -\lambda n \qquad \text{with initial condition } n(0) \text{ is known}}$$

to solve using Laplace, rewrite

$$\frac{dn}{dt} + \lambda n = 0$$

1st order linear and separable

now take the Laplace transform of both sides

$$\mathcal{L}\left\{\frac{dn}{dt}\right\} + \lambda\,\mathcal{L}\{n\} = \mathcal{L}\{0\}$$

table says
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$sN(s) - n(0) + \lambda N(s) = 0$$

← this whole equation now contains no derivatives

DE → algebra

so solve for $N(s)$ algebraically

initial condition

$$sN(s) + \lambda N(s) = n(0)$$

$$N(s) = \frac{n(0)}{s+\lambda}$$

now transform back

$$n(t) = \mathcal{L}^{-1}\{N(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{n(0)}{s+\lambda}\right\}$$

$$\boxed{n(t) = n(0)\, e^{-\lambda t}}$$

basic idea : convert a system which is a function of time into a system which is a function of freevency

[Fourier : time $\rightarrow$ modes of vibration]