Section 7.2: Inverse Transforms and Derivatives

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the inverse Laplace transform is written $\mathcal{L}^{-1}$

so that if $F(s) = \mathcal{L}\{f(t)\}$

then $f(t) = \mathcal{L}^{-1}\{F(s)\}$

nice property of $\mathcal{L}^{-1}$: it's linear, so

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\,\mathcal{L}^{-1}\{F(s)\} + b\,\mathcal{L}^{-1}\{G(s)\}$$

$$= a f(t) + b g(t)$$

how to find? use table

example: find the following inverse Laplace transforms

make into the form of a table entry

① $$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= \frac{1}{6}t^3$$

② $$\mathcal{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^3} - \frac{1}{s+8}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+8}\right\}$$

$$= 4 + 3t^2 - e^{-8t}$$

③ $\mathcal{L}^{-1}\left\{\frac{1}{7s-2}\right\} = \frac{1}{7}\mathcal{L}^{-1}\left\{\frac{1}{s-2/7}\right\} = \frac{1}{7}e^{2/7\,t}$

④ $\mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1/4}\right\}$

$\boxed{\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}}$

$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1/2}{s^2+1/4}\right\}$

$= \frac{1}{2}\sin\frac{t}{2}$

⑤ $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$

$= \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$

⑥ $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2-4s}\right\}$ $\Leftarrow$ need partial fractions

$\frac{s+3}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

$s+3 = A(s-4) + Bs$

let $s=0$: $\quad 3 = -4A \quad A = -3/4$

$s=4$: $\quad 7 = 4B \quad B = \frac{7}{4}$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{-\frac{3}{4s} + \frac{7}{4(s-4)}\right\}$$

$$= -\frac{3}{4} + \frac{7}{4}e^{4t}$$

(7) $$\mathcal{L}^{-1}\left\{\frac{12s^2+6}{s^4+s^2}\right\}$$

partial fractions:

$$\frac{12s^2+6}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$12s^2+6 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

so, let $s=0$:

$$6 = 0 + B + 0 \qquad \boxed{B=6}$$

$s=i$ so $s^2=-1$

$$-12+6 = -(Ci+D)$$

$$-6 + 0i = -Ci - D$$

so $\boxed{\begin{array}{l} C=0 \\ D=6 \end{array}}$

$s=1$ (doesn't make any term zero, but is easy to calculate)

$$18 = 2A + 2B + C + D$$

$$= 2A + 12 + 0 + 6$$

$$A = 0$$

so $$\mathcal{L}^{-1}\left\{\frac{12s^2+6}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s^2} + \frac{6}{s^2+1}\right\}$$

$$\ldots \frac{\;}{s^2(s^2+1)} \ldots \frac{\;}{s^2+1} \ldots$$

$$= 6\,\mathcal{L}^{-1}\left\{ \frac{1!}{s^2} \right\} + 6\,\mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\}$$

$$= 6t + 6\sin t$$

What about derivatives?

It turns out that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

why?

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

need parts

	D		I
	e^{-st}	$\oplus$	$f'(t)$
	$-se^{-st}$	$\ominus$	$f(t)$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\,\mathcal{L}\{f(t)\}$$

initial condition

$$= -f(0) + sF(s)$$

Similarly, $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

allow us to solve DEs if they are IVPs

example:

(1) solve $2\frac{dy}{dt} + y = 0$ if $y(0) = -3$

answer: $2\left[sY(s) - y(0)\right] + Y(s) = 0$

$$2sY(s) + 6 + Y(s) = 0$$

$$(2s+1)Y(s) = -6$$

$$Y(s) = \frac{-6}{2s+1}$$

$$= -\frac{6}{2}\frac{1}{s+\frac{1}{2}}$$

$$\boxed{y(t) = -3e^{-\frac{1}{2}t}}$$

(2) solve $y' - 2y = 3\cos 2t$ if $y(0) = 0$

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answer:

$$y' - 2y = 3\cos 2t, \qquad y(0) = 0$$

$$sY(s) - \cancel{y(0)}^{\,0} - 2Y(s) = \frac{3s}{s^2+4}$$

$$(s-2)Y(s) = \frac{3s}{s^2+4}$$

$$Y(s) = \frac{3s}{(s-2)(s^2+4)}$$

partial fractions

$$\frac{3s}{(s-2)(s^2-4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

$$3s = A(s^2+4) + (Bs+C)(s-2)$$

choose $s=2$:

$$6 = A \cdot 8 + 0 \qquad \text{so } A = \frac{3}{4}$$

choose $s = 2i$ (or choose $s=0$ and $s=1$ instead)

$$6i = 0 + (2Bi + C)(2i - 2)$$

$$6i = -4B - 4Bi + 2Ci - 2C$$

so $\begin{cases} 0 = -4B - 2C \\ 6i = -4Bi + 2Ci \end{cases} \Rightarrow \begin{cases} C = -2B \\ 3 = -2B + C \end{cases}$

so $C = 3/2$

$B = -3/4$

$$Y(s) = \frac{3s}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

want this

$$(s-2)(s^2+4) \qquad s-2 \qquad s^2+4$$

$$= \frac{3}{4(s-2)} - \frac{3}{4}\frac{s}{s^2+4} + \frac{3}{2(s^2+4)}$$

want this to look like $\frac{\omega}{s^2+\omega^2}$

$$= \frac{3}{4(s-2)} - \frac{3}{4}\frac{s}{s^2+4} + \frac{3}{4}\frac{2}{s^2+4}$$

$$\boxed{y(t) = \frac{3}{4}e^{2t} - \frac{3}{4}\cos 2t + \frac{3}{4}\sin 2t}$$

(3) Solve $y'' - 5y' + 6y = 4e^t$ where $y(0) = 3$, $y'(0) = 8$

apply Laplace

$$\left[s^2 Y(s) - s y(0) - y'(0)\right] - 5\left[sY(s) - y(0)\right] + 6Y(s) = \frac{4}{s-1}$$

now apply initial conditions

$$s^2 Y(s) - 3s - 8 - 5sY(s) + 15 + 6Y(s) = \frac{4}{s-1}$$

solve for $Y(s)$

$$(s^2 - 5s + 6)\,Y(s) = 3s - 7 + \frac{4}{s-1}$$

$$Y(s) = \frac{3s-7}{s^2-5s+6} + \frac{4}{(s^2-5s+6)(s-1)}$$

recommendation: combine into single fraction first, so don't have to do partial fractions twice

$$Y(s) = \frac{(3s-7)(s-1) + 4}{(s^2-5s+6)(s-1)}$$

$$= \frac{3s^2 - 10s + 11}{(s-1)(s-2)(s-3)}$$

partial fractions:

$$\frac{3s^2 - 10s + 11}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$3s^2 - 10s + 11 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

let $s=1$: $3 - 10 + 11 = 2A$ so $A = 2$

let $s=2$: $12 - 20 + 11 = -B$ $B = -3$

let $s=3$: $27 - 30 + 11 = 2C$ $C = 4$

so $Y(s) = \frac{2}{s-1} - \frac{3}{s-2} + \frac{4}{s-3}$

take inverse Laplace transform of both sides

$$y(t) = 2e^{t} - 3e^{2t} + 4e^{3t}$$