Section 7.3: Operational Properties I

Wednesday, March 22, 2023 11:29 AM

Laplace translations on s-axis, t-axis

translation on the s-axis

if $\mathcal{L}\{f(t)\} = F(s)$, then for "a" real,

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

why? (not tested)

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

this integral is identical to $F(s)$ except s is replaced by $s-a$

$$= F(s-a)$$

this is just $F(s)$ shifted to the right by "a" units

---

this means that

if $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, then $\mathcal{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$

if $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2+\omega^2}$, then $\mathcal{L}\{e^{at} \cos \omega t\} = \dfrac{s-a}{(s-a)^2+\omega^2}$

examples: find the following Laplace transforms examples: find the following Laplace transforms

① $\mathcal{L}\{e^{-3t}t^5\}$

$a=-3$

$f(t) = t^5$

$F(s) = \frac{5!}{s^6}$ ← ess

$= \frac{5!}{(s+3)^6}$

② $\mathcal{L}\{e^{-2t}\cos 4t\}$

$a=-2$

$f(t) = \cos 4t$

$F(s) = \frac{s}{s^2+16}$

$$\left.\begin{aligned} &= \frac{s+2}{(s+2)^2+16} \\ &= \frac{s+2}{s^2+4s+20} \end{aligned}\right\} \text{either}$$

example: find the following inverse Laplace transforms

① $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}$

$a=1$

$F(s) = \frac{1}{s^4} = \frac{1}{3!}\frac{3!}{s^4}$

$f(t) = \frac{1}{3!}t^3$

$= \frac{1}{3!}t^3e^t$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

denominator does not factor (if it did, you'd do partial fractions)

complete the square:

$$s^2+2s+5 = (s^2+2s+1)+5-1$$
$$= (s+1)^2+4$$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$

$a=-1$

$F(s) = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4}$

$f(t) = \frac{1}{2}\sin 2t$

$-t$

$f(t) = \frac{1}{2} \sin 2t$

$$= \frac{1}{2} e^{-t} \sin 2t$$

3 $\mathcal{L}^{-1}\left\{ \frac{2s+1}{s^2-6s+25} \right\}$

complete the square:

$s^2 - 6s + 25 = (s^2 - 6s + 9) + 25 - 9$

$= (s-3)^2 + 16$

$$= \mathcal{L}^{-1}\left\{ \frac{2s+1}{(s-3)^2+16} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{2(s-3)+1+6}{(s-3)^2+16} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{2(s-3)+7}{(s-3)^2+16} \right\}$$

$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+\omega^2} \right\} = \cos \omega t$

$\mathcal{L}^{-1}\left\{ \frac{\omega}{s^2+\omega^2} \right\} = \sin \omega t$

$$= 2\mathcal{L}^{-1}\left\{ \frac{s-3}{(s-3)^2+16} \right\} + \frac{7}{4}\mathcal{L}^{-1}\left\{ \frac{4}{(s-3)^2+16} \right\}$$

$$= 2e^{3t}\cos 4t + \frac{7}{4} e^{3t} \sin 4t$$

example: solve $y'' - 4y' + 4y = t^3 e^{2t}$

where $y(0) = y'(0) = 0$

take Laplace transform of both sides, then apply initial conditions

$$\left[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} \right] - 4\left[sY(s) - \cancel{y(0)}^{0} \right] + 4Y(s)$$

$$= \frac{3!}{(s-2)^4}$$

solve for $Y(s)$:

$$\left[s^2 - 4s + 4 \right] Y(s) = \frac{3!}{(s-2)^4}$$

$$\left[s^2 - 4s + 4\right] Y(s) = \frac{3!}{(s-2)^4}$$

$$(s-2)^2 \; Y(s) = \frac{3!}{(s-2)^4}$$

$$Y(s) = \frac{3!}{(s-2)^6} = \frac{3!}{5!} \; \frac{5!}{(s-2)^6}$$

apply $\mathcal{L}^{-1}$ to both sides

$$y(t) = \frac{3!}{5!} t^5 e^{2t}$$

$$\boxed{y(t) = \frac{1}{20} t^5 e^{2t}}$$

Section 7.3: cont'd

(2) solve $\quad 2y'' + 20y' + 51y = 0 \quad$ where $y(0) = 2, \; y'(0) = 0$

$$2\left[s^2 Y(s) - s\,\overset{2}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}}\right] + 20\left[sY(s) - \overset{2}{\cancel{y(0)}}\right] + 51\, Y(s) = 0$$

$$2s^2 Y(s) - 4s + 20s\,Y(s) - 40 + 51\, Y(s) = 0$$

$$2s^2 Y(s) + 20s\, Y(s) + 51\, Y(s) = 4s + 40$$

$$\left[2s^2 + 20s + 51\right] Y(s) = 4s + 40$$

$$Y(s) = \frac{4s + 40}{2s^2 + 20s + 51}$$

$$= \frac{4s + 40}{2(s+5)^2 + 1}$$

$$= \frac{2s + 20}{(s+5)^2 + \frac{1}{2}}$$

complete the square:

$$2s^2 + 20s + 51 = 2(s^2 + 10s + \underline{25}) + 51 - \underline{50}$$

$$= 2(s+5)^2 + 1$$

$$(s+5)^2 + \tfrac{1}{2}$$

aiming for: $\frac{s}{s^2+\omega^2}$ or $\frac{\omega}{s^2+\omega^2}$

$$= \frac{2(s+5) + 20 - 10}{(s+5)^2 + \tfrac{1}{2}}$$

$$Y(s) = \frac{2(s+5)}{(s+5)^2+\tfrac{1}{2}} + \frac{10\sqrt{2}\ \tfrac{1}{\sqrt{2}}}{(s+5)^2+\tfrac{1}{2}}$$

$$y(t) = 2e^{-5t}\cos\left(\frac{t}{\sqrt{2}}\right) + 10\sqrt{2}\,e^{-5t}\sin\left(\frac{t}{\sqrt{2}}\right)$$

cutoff for Test 3