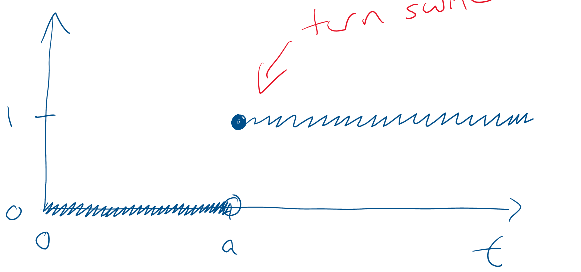
Section 7.3, Part II: Translations on the t-axis

Thursday, March 23, 2023 2:48 PM

frequently in electronics, you turn a switch on or off in your circuit

unit step function (Heaviside function)

turn switch on

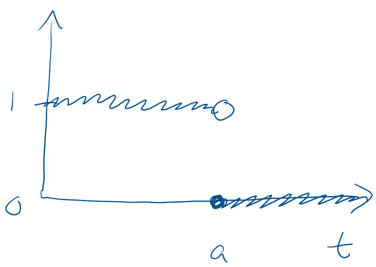


$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

note: Laplace is only defined for $t \ge 0$, so don't worry about $t < 0$

what about

$$f(t) = \begin{cases} 1 & 0 \le t < a \\ 0 & t \ge a \end{cases}$$



can rewrite $f(t)$ as

$$f(t) = 1 - \mathcal{U}(t-a)$$

why do we care? big idea is that piecewise functions can be written as a single function with the use of $\mathcal{U}$, the Heaviside function

examples:

① $f(t) = \begin{cases} 0 & 0 \le t < a \\ h(t) & t \ge a \end{cases}$

$= h(t)\,\mathcal{U}(t-a)$

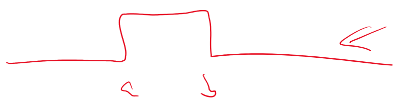
② $f(t) = \begin{cases} h(t) & 0 \le t < a \\ 0 & t \ge a \end{cases}$

$= h(t)\,[1 - \mathcal{U}(t-a)]$

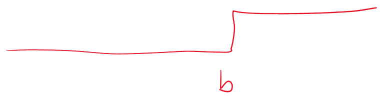
③ $f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$

$= g(t)\,[1 - \mathcal{U}(t-a)] + h(t)\,\mathcal{U}(t-a)$

④ $f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t \le b \\ 0 & t \ge b \end{cases}$



$= g(t)\,\mathcal{U}(t-a) - g(t)\,\mathcal{U}(t-b)$

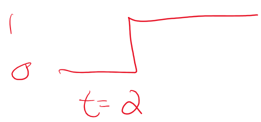


$= g(t)\,[\mathcal{U}(t-a) - \mathcal{U}(t-b)]$

examples: rewrite in terms of the Heaviside function:

① $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t^2 & t \ge 2 \end{cases}$

answer: $f(t) = t^2\,\mathcal{U}(t-2)$



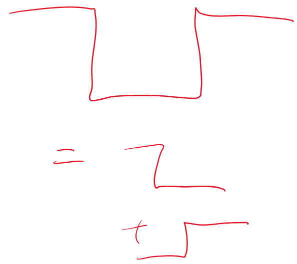
② $g(t) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$

answer: $g(t) = \sin t \left[1 - \mathcal{U}(t-2\pi)\right]$

(3) $h(t) = \begin{cases} e^{-t} & 0 \le t < \pi/4 \\ \sin t & t \ge \pi/4 \end{cases}$

answer: $h(t) = e^{-t}\left[1 - \mathcal{U}(t-\pi/4)\right] + \sin t \, \mathcal{U}(t-\pi/4)$

(4) $f(t) = \begin{cases} 8 & 0 \le t < 2 \\ 0 & 2 \le t < 3 \\ 8 & t \ge 3 \end{cases}$



answer:

$$f(t) = 8\left[1 - \mathcal{U}(t-2)\right] + 8\,\mathcal{U}(t-3)$$

So what is the Laplace transform of a Heaviside function

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s)$$

examples: evaluate

(1) $\mathcal{L}\{e^{2-t}\,\mathcal{U}(t-2)\}$

$$= \mathcal{L}\{e^{-(t-2)}\,\mathcal{U}(t-2)\}$$

$$= e^{-2s}\frac{1}{s+1}$$

$$\boxed{\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}}$$

Section 7.3 part 2: cont'd 2022/03/24

Section 7.3, part 2: cont'd 2023/03/24

two special cases of $\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s)$:

① $f(t)$ is a constant

note: if $f(t) = 1$
$F(s) = 1/s$

then $\mathcal{L}\{\underset{\substack{\uparrow \\ f(t)=1}}{}\ \mathcal{U}(t-a)\} = e^{-as}\cdot\frac{1}{s} = \frac{e^{-as}}{s}$ ⇐ table entry

② if $g(t) = f(t-a)$

then $g(t+a) = f(t)$

and $\mathcal{L}\{\underset{\text{unshifted}}{g(t)}\ \underset{\text{shifted}}{\mathcal{U}(t-a)}\} = e^{-as}F(s)$

$= e^{-as}\,\mathcal{L}\{f(t)\}$

$= \underset{\text{write this down}}{e^{-as}}\ \underbrace{\mathcal{L}\{g(t+a)\}}_{\text{evaluate this}}$

$$\boxed{\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}}$$

on Laplace sheet, it's on Line 3 under the table

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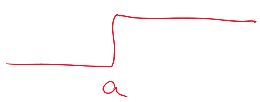
examples: find $\mathcal{L}\{f(t)\}$ if

$$y(t) = \begin{cases} 0 & \text{if } 0 \le t < 3 \\ t^2 & \text{if } t \ge 3 \end{cases}$$



answer:

$$y(t) = t^2\, u(t-3)$$

$$\mathcal{L}\{\underset{\substack{\uparrow \\ g(t)=t^2}}{t^2}\, u(t-3)\} = e^{-3s}\,\mathcal{L}\{(t+3)^2\}$$

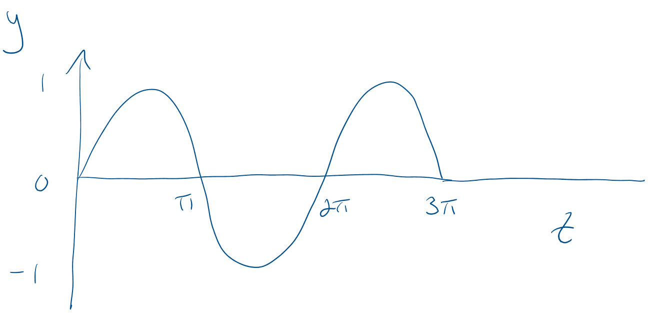
$$g(t+3) = (t+3)^2$$

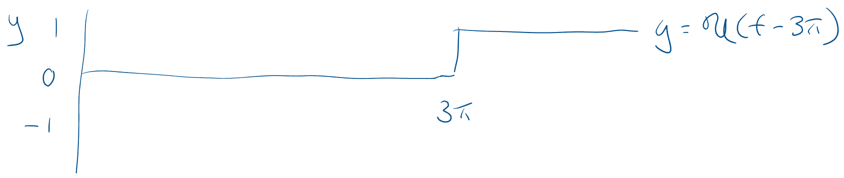
$$= e^{-3s}\,\mathcal{L}\{t^2 + 6t + 9\}$$

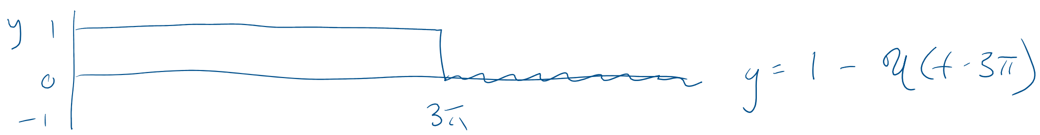
$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$$

example: find $\mathcal{L}\{y(t)\}$ if

$$y(t) = \begin{cases} \sin t & \text{if } 0 \le t < 3\pi \\ 0 & t \ge 3\pi \end{cases}$$





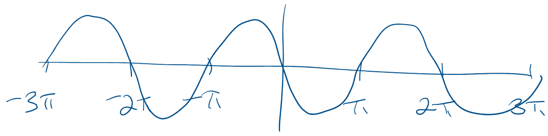


$$y(t) = \sin t\,[1 - u(t-3\pi)]$$

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t\; u(t-3\pi)\}$$

$$= \frac{1}{s^2+1} - e^{-3\pi s}\,\mathcal{L}\{\underbrace{\sin(t+3\pi)}_{\sin t \text{ shifted left}}\}$$

sin t shifted left by 3π



$$= \frac{1}{s^2+1} - e^{-3\pi s}\,\mathcal{L}\{-\sin t\}$$

$$= \frac{1}{s^2+1} + e^{-3\pi s}\,\frac{1}{s^2+1}$$

what about taking inverse Laplace transforms with Heaviside functions?

recall: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$

example: find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-4}\right\}$

answer: $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-4}\right\}$

e^{-3s} will give us a factor of $\mathcal{U}(t-3)$

now take $F(s)$, find $f(t)$, and shift to get $f(t-a)$

$F(s) = \dfrac{1}{s-4}$

gives $f(t) = e^{4t}$

$f(t-3) = e^{4(t-3)}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s-4}\right\} = e^{4(t-3)}\,\mathcal{U}(t-3)$$

example: find $\mathcal{L}^{-1}\left\{\dfrac{e^{-4s}}{\ }\right\} = \dfrac{(t-4)^7}{\ }\,\mathcal{U}(t-4)$

example: find $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^8}\right\} = \frac{(t-4)^7}{7!}\mathcal{U}(t-4)$

examples:

① solve $y' + y = f(t)$ where $y(0) = 0$

and $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & t \ge 1 \end{cases}$

answer: $f(t) = 1\left[1 - \mathcal{U}(t-1)\right] + (-1)\,\mathcal{U}(t-1)$

$= 1 - 2\,\mathcal{U}(t-1)$

so DE is now $y' + y = 1 - 2\,\mathcal{U}(t-1)$

$$\left[sY(s) - \cancel{y(0)}^{\,0}\right] + Y(s) = \frac{1}{s} - \frac{2e^{-s}}{s}$$

$$(s+1)\,Y(s) = \frac{1-2e^{-s}}{s}$$

$$Y(s) = \frac{1-2e^{-s}}{s(s+1)}$$

$$= (1-2e^{-s})\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{2e^{-s}}{s} + \frac{2e^{-s}}{s+1}$$

partial fractions

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$\boxed{y(t) = 1 - e^{-t} - 2\,\mathcal{U}(t-1) + 2e^{-(t-1)}\,\mathcal{U}(t-1)}$$

Section 7.3: cont'd 2023/03/28

(2) solve $y'' - 5y' + 6y = \mathcal{U}(t-1)$, $y(0) = 0$, $y'(0) = 1$

answer:

$$\left[s^2 Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,1}\right] - 5\left[sY(s) - \cancel{y(0)}^{\,0}\right] + 6Y(s) = \frac{e^{-s}}{s}$$

why?

4th line under table

$$\mathcal{L}\{f(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\{\underset{\uparrow}{}\,\mathcal{U}(t-1)\} = \frac{e^{-s}}{s}$$

$f(t) = 1$

but the quicker route is to notice

$$\mathcal{U}(t-a) \to \frac{e^{-as}}{s}$$

$$(s^2 - 5s + 6)\,Y(s) = \frac{e^{-s}}{s} + 1$$

$$Y(s) = \frac{e^{-s}}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$$

partial fractions:

$$\frac{1}{s(s-2)(s-3)} = \frac{1}{6s} - \frac{1}{2(s-2)} + \frac{1}{3(s-3)}$$

$$\frac{1}{(s-2)(s-3)} = \frac{-1}{s-2} + \frac{1}{s-3}$$

$$Y(s) = e^{-s}\left[\frac{1}{6s} - \frac{1}{2(s-2)} + \frac{1}{3(s-3)}\right] - \frac{1}{s-2} + \frac{1}{s-3}$$

$$y(t) = \mathcal{U}(t-1)\left[\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right] - e^{2t} + e^{3t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} = 1\,\mathcal{U}(t-1)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{6s}\right\} = \frac{1}{6}\,\mathcal{U}(t-1)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{2(s-2)}\right\} = \frac{1}{2}e^{2(t-1)}\,\mathcal{U}(t-1) \qquad \mathcal{L}^{-1}\left\{\frac{1}{2(s-2)}\right\} = \frac{1}{2}e^{2t}$$

will give
Heaviside and
a shift

on table:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$$