Section 7.4: Operational Properties II

Wednesday, April 1, 2020 8:29 PM

derivatives of transforms:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

why? we'll show the first derivative:

suppose $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \left(\frac{d}{ds} e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty \left(-t e^{-st}\right) f(t)\,dt$$

$$= -\int_0^\infty e^{-st} \left[t f(t)\right] dt$$

$$= -\mathcal{L}\{t f(t)\}$$

Why do we care?

recall: $$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

then $$\mathcal{L}\{t \sin \omega t\} = (-1)\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= -1\,(-1)\frac{\omega(2s)}{(s^2+\omega^2)^2}$$

$$= \frac{2\omega s}{(s^2+\omega^2)^2}$$

similarly

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$$

then $$\mathcal{L}\{t\cos \omega t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

$$= (-1)\frac{(s^2+\omega^2)\cdot 1 - s(2s)}{(s^2+\omega^2)^2}$$

$$= (-1)\frac{s^2+\omega^2-2s^2}{(s^2+\omega^2)^2}$$

$$= (-1)\frac{\omega^2-s^2}{(s^2+\omega^2)^2}$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

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using this rule,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d}{ds^n} F(s)$$

we find that

$$\mathcal{L}\{te^{7t}\} = (-1)\frac{d}{ds}\left[\mathcal{L}\{e^{7t}\}\right]$$

$$= -\frac{d}{ds}\left(\frac{1}{s-7}\right)$$

$$= +\frac{1}{(s-7)^2}$$

but previously, we found that

$$\mathcal{L}\{te^{7t}\} = \frac{1}{(s-7)^2}$$

same result!

$\mathcal{L}\{t\} = \frac{1}{s^2}$

the e^{7t} shift $s \to s-7$

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example: solve $y' - y = te^t \sin t$, $y(0) = 0$

now $\mathcal{L}\{t \sin t\} = \dfrac{2s}{(s^2+1)^2}$

and e^t makes $s \to s-1$

so $\mathcal{L}\{te^t \sin t\} = \dfrac{2(s-1)}{}$

so $\mathcal{L}\{te^t \sin t\} = \dfrac{2(s-1)}{((s-1)^2+1)^2}$

back to DE:

$$y' - y = te^t \sin t, \quad y(0)=1$$

$$[sY(s) - \cancel{y(0)}^{\,0}] - Y(s) = \frac{2(s-1)}{((s-1)^2+1)^2}$$

$$\cancel{(s-1)}\,Y(s) = \frac{2\cancel{(s-1)}}{((s-1)^2+1)^2}$$

$$Y(s) = \frac{2}{((s-1)^2+1)^2}$$

now $s-1$ gives factor of e^t from shift

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+w^2)^2}\right\} = \frac{\sin wt - wt\cos wt}{2w^3}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{((s-1)^2+1)^2}\right\} = \cancel{2}e^t\left[\frac{\sin t - t\cos t}{\cancel{2}}\right]$$

$$y(t) = e^t[\sin t - t\cos t]$$