Section 7.4: Operational Properties II - derivatives of transforms

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derivatives of transforms:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

why? digression: (not tested)

we'll show the first derivative

say $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

now $\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st} \left[t f(t)\right] dt$$

$$= -\mathcal{L}\{t f(t)\}$$

so $\mathcal{L}\{t \sin \omega t\} = -1 \frac{d}{ds} \mathcal{L}\{\sin \omega t\}$

$$= -1 \frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= -1(-1)\frac{2s\omega}{(s^2+\omega^2)^2} = \frac{2s\omega}{(s^2+\omega^2)^2}$$

but we've actually provided an entry in the table for this, so you can go directly to

$f(t)$	$F(s)$
$t \sin \omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
$t \cos \omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

so let's look at

$$\mathcal{L}\{t e^{7t}\}$$

- three different ways to go

① $\mathcal{L}\{t e^{7t}\} = -1 \dfrac{d}{ds} \mathcal{L}\{e^{7t}\}$

$$= -1 \frac{d}{ds}\left(\frac{1}{s-7}\right)$$

$$= -1 \frac{-1}{(s-7)^2} = \frac{1}{(s-7)^2}$$

② $\mathcal{L}\{t e^{7t}\} = \dfrac{1}{(s-7)^2}$

$\mathcal{L}\{t\} = \dfrac{1}{s^2}$

the e^{7t} shift

$s \to s-7$

③ everyone's favourite - table

$f(t)$	$F(s)$

$$\frac{e^{at}t^n}{n!} \qquad \frac{1}{(s-a)^{n+1}}$$

$$\mathcal{L}\{te^{7t}\} = \mathcal{L}\left\{\frac{t^1 e^{7t}}{1!}\right\} = \frac{1}{(s-7)^{1+1}} = \frac{1}{(s-7)^2}$$

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example: solve $y' - y = te^t \sin t$, $y(0) = 0$

answer: think about the $te^t \sin t$ term as

$$e^t \underbrace{(t \sin t)}_{\text{this has an entry in the table}}$$

e^t make $s \to s-1$

from

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$f(t)$	$F(s)$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$t \sin t$	$\frac{2s}{(s^2+1)^2}$

so replace s by $s-1$:

$$\mathcal{L}\{e^t t \sin t\} = \frac{2(s-1)}{((s-1)^2+1)^2}$$

back to our DE:

$$y' - y = te^t \sin t, \quad y(0) = 0$$

$$sY(s) - \cancel{y(0)}^{0} - Y(s) = \frac{2(s-1)}{((s-1)^2+1)^2}$$

$$sY(s) - \cancel{y(0)} - Y(s) = \frac{2(s-1)}{((s-1)^2+1)^2}$$

$$\cancel{(s-1)}\,Y(s) = \frac{2\cancel{(s-1)}}{((s-1)^2+1)^2}$$

$$Y(s) = \frac{2}{((s-1)^2+1)^2}$$

looking at the table, what entry is in the right form?

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{\sin\omega t - \omega t\cos\omega t}{2\omega^3}$$

so, how to deal with the $s-1$?

recall we said e^t makes $s \to s-1$

so the $s-1$ in our term above will give a factor of e^t in our answer

$$y(t) = 2e^t\left[\frac{\sin t - t\cos t}{2}\right]$$

$$= e^t(\sin t - t\cos t)$$

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we will omit convolutions

(how you take the inverse transform of a product)