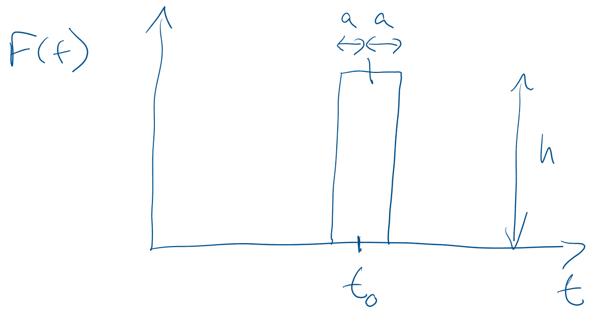
Section 7.5: The Dirac Delta Function

Wednesday, March 29, 2023 12:01 PM

impulse: suppose you wish to model hitting a golf ball with a club or tapping a mass on a spring with your finger



t_0 is the time at which you hit the golf ball

for a unit impulse, want the area to be equal to one, so $h = \frac{1}{2a}$

<u>but</u> if you want the impulse to be instantaneous, then take the limit as $a \to 0$

Dirac delta function:

$$\delta(t - t_0) = \lim_{a \to 0} F(t)$$

Greek lower-case delta

$$= \begin{cases} \infty & \text{at } t = t_0 \\ 0 & \text{at } t \neq t_0 \end{cases}$$

where $\int_0^\infty \delta(t - t_0)\, dt = 1$

stated without proof:

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$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\} = 1$$

Section 7.5 cont'd 2023/03/30

digression: nice property $\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)$

example: solve $y' + y = \delta(t-1)$, $y(0) = 2$

answer: $[sY(s) - \cancel{y(0)}^{2}] + Y(s) = e^{-s}$

$$(s+1)\,Y(s) = 2 + e^{-s}$$

$$Y(s) = \frac{2}{s+1} + \frac{e^{-s}}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} = 2e^{-t}$$

want $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$$

$$\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s+1}\right\} = e^{-(t-1)}\,\mathcal{U}(t-1)$$

$F(s) = 1/s+1$

$f(t) = e^{-t}$

$$y(t) = \underbrace{2e^{-t}}_{\text{homogeneous}} + \underbrace{e^{-(t-1)}\,\mathcal{U}(t-1)}_{\text{response to impulse}}$$

homogeneous solution   response to impulse

example: solve the following, giving a piecewise solution

$$y'' + 16y = \delta(t - 2\pi), \quad y(0) = 0, \; y'(0) = 0$$

answer:

$$\left[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right] + 16\,Y(s) = e^{-2\pi s}$$

$$(s^2 + 16)\,Y(s) = e^{-2\pi s}$$

$$Y(s) = \frac{e^{-2\pi s}}{s^2 + 16}$$

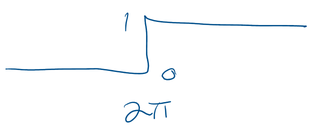
$$= \frac{e^{-2\pi s}}{4} \, \frac{4}{s^2 + 16}$$

$e^{-2\pi s}$ shifts $t \to (t - 2\pi)$

and multiplies by $\mathcal{U}(t - 2\pi)$

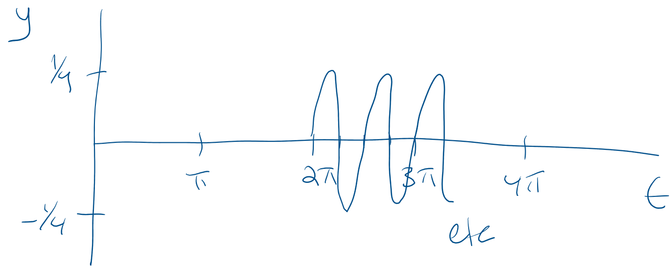
$$y(t) = \frac{1}{4} \underbrace{\sin 4(t - 2\pi)}_{\text{periodic}} \; \mathcal{U}(t - 2\pi)$$

$$y(t) = \frac{1}{4} \sin 4t \; \mathcal{U}(t - 2\pi)$$

$\mathcal{U}(t)$ 

$$= \begin{cases} 0 & 0 \le t < 2\pi \\ \frac{1}{4} \sin 4t & t \ge 2\pi \end{cases}$$

what does this look like?


y
1/4
-1/4
π
2π
3π
4π
t
etc