Section 8.1: Systems of Linear First-Order DEs

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a system of linear first-order DEs can be written in the form

$$\vec{X}'(t) = A\vec{X}(t) + \vec{F}(t)$$

where A is an $N \times N$ matrix
$\vec{X}(t)$ and $\vec{F}(t)$ are column vectors in $\mathbb{R}^N$

note: system is homogeneous if $\vec{F}(t) = 0$

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example: rewrite the following system of DEs in matrix form:

$$\begin{cases} \dfrac{dx}{dt} = 4x - 7y \\ \dfrac{dy}{dt} = 5x \end{cases}$$

answer:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$X'(t) = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}\vec{X}$$

$$\begin{bmatrix} 5 & 0 \end{bmatrix}$$

where $\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$

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similarly,

$$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = x + 2z \\ \dfrac{dz}{dt} = -x + z \end{cases}$$

is equivalent to $\vec{X}' = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \vec{X}$

$$\begin{cases} \dfrac{dx}{dt} = 4x + 2y + e^t \\ \dfrac{dy}{dt} = -x + 3y - e^t \end{cases}$$

answer: $\vec{X}' = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

or $\begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

example: verify that $\vec{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t}$ is a solution

to $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

answer: $\vec{X} = \begin{bmatrix} e^{-5t} \\ 2e^{-5t} \end{bmatrix}$

$\vec{X}' = \begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix}$

now sub back into system of DEs:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix} = \overset{2\times2}{\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}}\overset{2\times1}{\begin{bmatrix} e^{-5t} \\ 2e^{-5t} \end{bmatrix}}$$

$$= \overset{2\times1}{\begin{bmatrix} 3e^{-5t} - 8e^{-5t} \\ 4e^{-5t} - 14e^{-5t} \end{bmatrix}}$$

$$\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix} \checkmark$$

preview: where does this solution come from?

$$\vec{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t} \qquad A = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}$$

the eigenvalues and eigenvectors of A are

$$\lambda_1 = 1 \quad \text{with} \quad \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and $\lambda_2 = -5$ with $\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$