Section 8.1: Systems of Linear First Order DEs

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example: rewrite the following systems of DEs into matrix form:

a) $$\begin{cases} \dfrac{dx}{dt} = 4x - 7y \\ \dfrac{dy}{dt} = 5x \end{cases}$$

answer:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{X}' = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}\vec{X}$$

where $\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$

b) $$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = x + 2z \\ \dfrac{dz}{dt} = -x + z \end{cases}$$

is equivalent to

$$\vec{X}' = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}\vec{X}$$

c) $$\begin{cases} \dfrac{dx}{dt} = 4x + 2y + e^{t} \\ \dfrac{dy}{dt} = -x + 3y - e^{t} \end{cases}$$

matrix form: $\vec{X}' = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}\vec{X} + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$

matrix form: $\vec{X}' = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

or $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

So a system of linear first order DEs can be written in the form

$$\vec{X}' = A\vec{X} + \vec{F}(t)$$

an $N \times N$ matrix

$\vec{X}$ and $\vec{F}(t)$ are column vectors in $\mathbb{R}^N$

note: system is homogeneous if $F(t) = 0$

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example: verify that $\vec{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t}$ is a solution to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

answer: if $\vec{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t} = \begin{bmatrix} e^{-5t} \\ 2e^{-5t} \end{bmatrix}$

then $\vec{X}' = \begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix}$

now sub back into DE:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}}^{2\times 2}\overbrace{\begin{bmatrix} e^{-5t} \\ 2e^{-5t} \end{bmatrix}}^{2\times 1}$$

$$\begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 2e^{-5t} \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-5t} - 8e^{-5t} \\ 4e^{-5t} - 14e^{-5t} \end{bmatrix}$$

$$= \begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix}$$

preview: where does this solution come from?

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t} \qquad A = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}$$

the eigenvalues and eigenvectors of A are:

$$\lambda_1 = 1 \quad \text{with} \quad \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5 \quad \text{with} \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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review of eigenvectors and eigenvalues:

example: for matrix $A = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}$, find the eigenvalues and associated eigenvectors

answer: find eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = 0$$

$$0 = (3-\lambda)(-7-\lambda) - 4(-4)$$

$$= -21 + 7\lambda - 3\lambda + \lambda^2 + 16$$

$$0 = (3-\lambda)(-7-\lambda) + 4(4)$$

$$= -21 + 7\lambda - 3\lambda + \lambda^2 + 16$$

$$= \lambda^2 + 4\lambda - 5$$

$$= (\lambda - 1)(\lambda + 5)$$

$$\lambda = -5, 1$$

for $\lambda_1 = 1$

$$A - \lambda I = 0$$

$$\left[\begin{array}{cc|c} 3-\lambda & -4 & 0 \\ 4 & -7-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 4 & -8 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free variable

let $y = t$

$x - 2y = 0$

$x = 2y$

$x = 2t$

solution is $x = 2t$, $y = t$ $\qquad \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

now $\lambda_2 = -5$

$$A - \lambda I = 0$$

$$\left[\begin{array}{cc|c} 3-\lambda & -4 & 0 \\ 4 & -7-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 8 & -4 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $y = t$

$$x - \tfrac{1}{2}y = 0$$

$$x = \tfrac{1}{2}y = \tfrac{1}{2}t$$

$$\vec{x}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$