Section 8.2: Solving Homogeneous Systems

Tuesday, April 04, 2023 12:06 PM

solve $\vec{X}' = A\vec{X}$ where A is an $N \times N$ matrix

$\vec{X}$ is a column vector in $\mathbb{R}^N$

the general solution is

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3 + \dots \; c_N\vec{X}_N$$

where $\vec{X}_1, \vec{X}_2, \dots \vec{X}_N$ are N linearly independent solutions

method of solution:

we look for $\vec{X} = \vec{k}e^{\lambda t}$

for a vector $\vec{k} \neq \vec{0}$ and some number λ

but if $\vec{X} = \vec{k}\,e^{\lambda t}$ where k is not a function of t

then $\vec{X}' = \lambda\vec{k}e^{\lambda t}$

so plus into

$$\vec{X}' = A\vec{X}$$

$$\lambda\vec{k}e^{\lambda t} = A\vec{k}e^{\lambda t}$$

$$\lambda\vec{k} = A\vec{k}$$

so λ = eigenvalues of A

$\vec{k}$ = associated eigenvectors of A

So: 4 cases:

1. distinct real eigenvalues
2. repeated λ with enough eigenvectors

③ " " " not enough eigenvectors

④ complex λ

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① distinct real eigenvalues:

example: solve $\begin{cases} \frac{dx}{dt} = 2x + 2y \\ \frac{dy}{dt} = x + 3y \end{cases}$ with $\vec{X}(0) = \begin{bmatrix} -18 \\ 3 \end{bmatrix}$

answer: $\vec{X}' = \underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \vec{X}$

$$\det(A - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$0 = (2-\lambda)(3-\lambda) - 2$$

$$0 = 6 - 5\lambda + \lambda^2 - 2$$

$$0 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

$$\lambda = 1, 4$$

then $(A - \lambda I)\vec{k} = 0$

let $\lambda_1 = 1$

$$\left[\begin{array}{cc|c} 2-\lambda & 2 & 0 \\ 1 & 3-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $y = t$

$x + 2y = 0$
so
$x = -2y$
$= -2t$

$$\begin{cases} x = -2t \\ y = t \end{cases} \quad \text{so } \vec{k} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} x = -2t \\ y = t \end{cases} \qquad \text{so } \vec{k} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

let $\lambda_2 = 4$

$$\left[\begin{array}{cc|c} 2-\lambda & 2 & 0 \\ 1 & 3-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftarrow x - y = 0$$

$x = y$

$= t$

let $y = t$

$$\begin{cases} x = t \\ y = t \end{cases} \qquad \text{so } \vec{k}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so now $\vec{X}(t) = c_1 \vec{k}_1 e^{\lambda_1 t} + c_2 \vec{k}_2 e^{\lambda_2 t}$

$$= c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

now: initial conditions $\vec{X}(0) = \begin{bmatrix} -18 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} -18 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} -2c_1 + c_2 &= -18 \\ c_1 + c_2 &= 3 \end{aligned}$$

solve to get $c_1 = 7$, $c_2 = -4$

then $$\boxed{\vec{X}(t) = \begin{bmatrix} -14 \\ 7 \end{bmatrix} e^{t} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} e^{4t}}$$

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② repeated λ with enough eigenvectors solve

$$\vec{x}' = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

answer: $\det(A - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

this is upper triangular so determinant is product of main diagonal (true also for lower triangular and diagonal matrices)

$$= (2-\lambda)(2-\lambda)(1-\lambda)$$

$$\lambda = 1, \underbrace{2, 2}$$

alg. mult of $\lambda = 2$ is 2

so let $\lambda_1 = 1$ $\quad (A - \lambda I)\vec{k} = 0$

already in RREF $\rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x + 3z = 0$ $\quad x = -3t$

$y = 0$

z is free

let $z = t$

$$\begin{cases} x = -3t \\ y = 0 \\ z = t \end{cases} \qquad \vec{k}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

and let $\lambda_2 = 2$

$$(A - \lambda I)\vec{k} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = 0$

let $x = s$ $\quad$ let $y = t$

$x = s \qquad y = t$

$$\begin{cases} x = s \\ y = t \\ z = 0 \end{cases} \qquad \vec{k} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t$$

so eigenvectors for $\lambda_2 = 2$

are $\vec{k}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{k}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

note: algebraic multiplicity = geometric multiplicity

number of times the term appears in $\det(A - \lambda I)$ expression

how many eigenvectors do you get for that eigenvalue

so $A = 3 \times 3$ so we need 3 eigenvectors

and we <u>have</u> 3 eigenvectors

so go ahead and write solution

$$\vec{X}(t) = \sum c_i \vec{k}_i e^{\lambda_i t}$$

$$\boxed{\vec{X}(t) = c_1 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}}$$

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(3) repeated eigenvalues but not enough eigenvectors

what do the solutions look like?

$\vec{X}_1$ is same as before $= \vec{k}_1 e^{\lambda_1 t}$

but what's $\vec{x}_2$?

$$\boxed{\vec{x}_2 = (\vec{k}_1 t + \vec{p}) e^{\lambda t} \quad \text{where} \quad (A - \lambda I)\vec{p} = \vec{k}_1}$$

and if necessary,

$$\vec{x}_3 = \left(\vec{k}_1 \frac{t^2}{2} + \vec{p} t + \vec{Q}\right) e^{\lambda t}$$

where $(A - \lambda I)\vec{Q} = \vec{p}$

example: solve

$$\begin{cases} \dfrac{dx}{dt} = -6x + 5y \\[2ex] \dfrac{dy}{dt} = -5x + 4y \end{cases}$$

answer: $A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = 0 = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$$

$$= (-6-\lambda)(4-\lambda) + 25$$

$$= -24 + 2\lambda + \lambda^2 + 25$$

$$0 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$$

$$\lambda = -1, -1$$

repeated

let $\lambda = -1$

$$(A - \lambda I)\vec{k} = 0$$

$$\left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x - y = 0$

so $x = t$

let $y = t$

$$\begin{cases} x = t \\ y = t \end{cases} \qquad \vec{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

geo mult = 1

geo mult < algebraic

not enough eigenvectors!

then $\vec{x}_1 = \vec{k} e^{\lambda t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

but what is $\vec{x}_2$?

$$\vec{x}_2 = (\vec{k}_1 t + \vec{p}) e^{\lambda t}$$

where $(A - \lambda I)\vec{p} = \vec{k}_1$

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $-5p_1 + 5p_2 = 1$ ← now choose <u>any</u> non-zero $\vec{p}$ that satisfies this

so let $p_1 = 0$, to get $p_2 = 1/5$

$p = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$ ← not unique - many different $\vec{p}$s that will work

so $\vec{x}_2 = (\vec{k} t + p) e^{\lambda t}$

$$= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \right) e^{-t}$$

and finally

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \right) e^{-t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \right) e^{-t}$$

$$= e^{-t} \left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \right)$$

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(4) complex eigenvalues

- same idea as 2 distinct real eigenvalues but with

$$\lambda = \alpha \pm \beta i$$

so $\lambda_1 = \alpha + \beta i$

$\lambda_2 = \alpha - \beta i$ $\Leftarrow$ complex conjugate of λ_1

note:
$\lambda_1 + \lambda_2 = 2\alpha$ = real number
$\lambda_1 - \lambda_2 = 2\beta i$ = pure imaginary

and λ_1 has eigenvector $\vec{k}_1$

λ_2 $\vec{k}_2$ where $\vec{k}_2$ is conjugate of $\vec{k}_1$

so could say

$$\vec{X} = c_1 \underbrace{\vec{k}_1 e^{\lambda_1 t}}_{z_1} + c_2 \underbrace{\vec{k}_2 e^{\lambda_2 t}}_{z_2}$$

but this is ugly because you have complex numbers everywhere

so instead, use similar argument to 2^{nd} order linear DEs with constant coeffs:

if $\vec{k}_1 = \underbrace{\vec{A}}_{\text{real}} + \underbrace{i\vec{B}}_{\text{imaginary}}$, $\vec{z} = (\vec{A} + i\vec{B})e^{(\alpha+\beta i)t}$

$$= (\vec{A} + i\vec{B})e^{\alpha t}e^{i\beta t}$$

$$= e^{\alpha t}(\vec{A} + i\vec{B})(\cos\beta t + i\sin\beta t)$$

let $X_1 = Re(\vec{z}_1) = e^{\alpha t}(\vec{A}\cos\beta t - \vec{B}\sin\beta t)$

$\vec{X}_2 = \underbrace{Im(\vec{z}_1)}$ $= e^{\alpha t}(\vec{A}\sin\beta t + \vec{B}\cos\beta t)$

ie $z_1 + z_2$ is real

$z_1 - z_2$ is imaginary

so solution vectors $\vec{X}_1$ and $\vec{X}_2$ can be written as

$$\vec{X}_1 = e^{\alpha t}(\vec{A}\cos\beta t - \vec{B}\sin\beta t)$$

$$\vec{X}_2 = e^{\alpha t}(\vec{A}\sin\beta t + \vec{B}\cos\beta t)$$

where $\vec{A} = Re(\vec{k}_1)$

$\vec{B} = Im(\vec{k}_1)$

example: solve $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = -2x - y \end{cases}$

answer: $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) + 2 = 0$$

$$-1 + \lambda^2 + 2 = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

note: $\lambda = \alpha \pm \beta i$

here $\alpha = 0$
$\beta = 1$

let $\lambda_1 = i$

$$(A - \lambda I)\vec{k} = 0$$

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

swap rows, then divide top row by -2, second row will be all zeros

$\Downarrow$ RREF

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

y is free variable
let $y = t$

$x + \frac{1}{2}(1+i)y = 0$

$x = -\frac{1}{2}(1+i)y$

$= -\frac{1}{2}(1+i)t$

$$\begin{cases} x = -\frac{1}{2}(1+i)t \\ y = t \end{cases}$$

$$\vec{k}_1 = \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

$\Downarrow$ scale

$$\vec{k}_1 = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

(note: $\lambda_2 = -i$ and $\vec{K}_2 = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$)

so $\vec{K}_1 = \begin{bmatrix} -1-i \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\vec{A}} + i \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\vec{B}}$

because $\vec{A} = Re(\vec{K}_1)$

and $\vec{X}_1 = \vec{A}\cos\beta t - \vec{B}\sin\beta t$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t$$

$\vec{X}_2 = \vec{A}\sin\beta t + \vec{B}\cos\beta t$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}\sin t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t$$

finally general solution is $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2$

$$\vec{X} = c_1\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\sin t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t\right)$$

acceptable answer

$$= c_1\begin{bmatrix} \sin t - \cos t \\ 2\cos t \end{bmatrix} + c_2\begin{bmatrix} -\sin t - \cos t \\ 2\sin t \end{bmatrix}$$