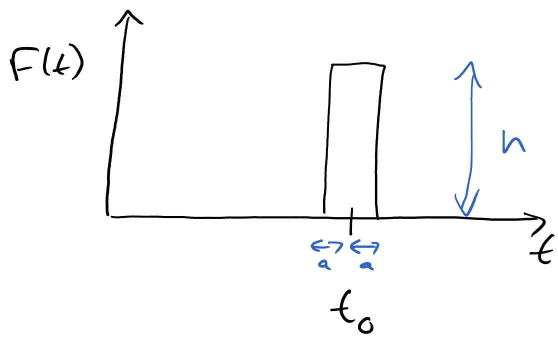
# Section 7.5: The Dirac Delta Function

Wednesday, November 6, 2019 1:47 PM

impulse: suppose you wish to model hitting a golf ball, or tapping a mass on a spring with your finger




for unit impulse, want area to be one, so $h = \frac{1}{2a}$

but if you want the impulse to be instantaneous, then take the limit as $a \to 0$:

Dirac delta function

$$\delta(t - t_0) = \lim_{a \to 0} F(t)$$

Greek lower-case delta

$$= \begin{cases} \infty & \text{at } t = t_0 \\ 0 & \text{at } t \neq t_0 \end{cases}$$

where $\int_0^\infty \delta(t - t_0)\, dt = 1$

stated without proof: $\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$

$$\mathcal{L}\{\delta(t)\} = 1$$

nice property: $\displaystyle\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)$

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example: solve $y' + y = \delta(t-1)$, $y(0) = 2$

$$sY(s) - \cancel{y(0)}^{\,2} + Y(s) = e^{-s}$$

$$(s+1)\,Y(s) = 2 + e^{-s}$$

$$Y(s) = \frac{2}{s+1} + \frac{e^{-s}}{s+1}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$$

and $\displaystyle\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$y(t) = 2e^{-t} + \underbrace{e^{-(t-1)}\,\mathcal{U}(t-1)}_{\text{response to impulse}}$$

homogeneous solution

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example: solve the following, giving a piecewise solution

$$y'' + 16y = \delta(t-2\pi), \quad y(0) = 0, \; y'(0) = 0$$

$$\left[s^2 Y(s) - \cancelto{0}{s y(0)} - \cancelto{0}{y'(0)}\right] + 16\,Y(s) = e^{-2\pi s}$$

$$(s^2+16)\,Y(s) = e^{-2\pi s}$$

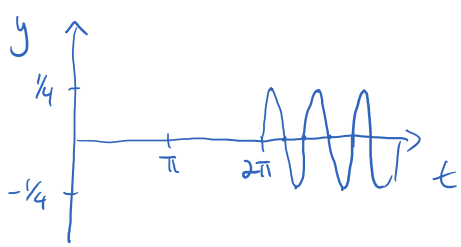
$$Y(s) = \frac{e^{-2\pi s}}{s^2+16} = \frac{1}{4} e^{-2\pi s}\left(\frac{4}{s^2+16}\right)$$

the $e^{-2\pi s}$ shifts $t \to t-2\pi$
and multiplies by $\mathcal{U}(t-2\pi)$

$$y(t) = \frac{1}{4} \underbrace{\sin 4(t-2\pi)}_{\text{periodic}} \mathcal{U}(t-2\pi)$$

$$y(t) = \frac{1}{4} \sin 4t \; \mathcal{U}(t-2\pi)$$

$$y(t) = \begin{cases} 0 & 0 \le t < 2\pi \\ \frac{1}{4}\sin 4t & t \ge 2\pi \end{cases}$$




So, a mass on spring starts at rest, gets tapped at $t = 2\pi$ and starts oscillating